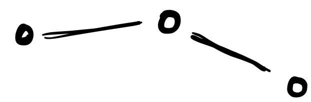
Relative Distance Formations



$$D = \{ \|x_i - x_j\| = d_{ij} \mid d_{ij} > 0 \}$$

$$\rightarrow \|x_i - x_j\|^2 = (x_i - x_j)^T (x_i - x_j)$$

Infinitesimal Rigidity:

"what (infinitesimal) motions are consistent w/ formation?"

CONDITION: $2(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0$

$\dot{x}_i = u_i$ — motion of node i

$\dot{x}_j = u_j$ — motion of node j

$$(x_i - x_j)^T (u_i - u_j) = 0 \quad \forall e \in (i,j)$$

DEFINE FORMATION STRUCTURE IN TERMS OF G w/ an incidence matrix D.

position of all agents $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{np}$ $\quad p = 2, 3 \quad$ $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^{np}$

$$[x_1 \cdots x_n] \begin{bmatrix} I & & \\ -I & I & \\ & -I & \end{bmatrix} \quad \begin{bmatrix} I & -I & \\ & I & -I \\ & \vdots & \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$x^T (D \otimes I)(D^T \otimes I) u = \sum_{e=(i,j)} (x_i - x_j)'(u_i - u_j) = 0$$

$$\begin{bmatrix} x_1^T - x_2^T & x_2^T - x_3^T & \cdots \end{bmatrix} \begin{bmatrix} u_1 - u_2 \\ u_2 - u_3 \\ \vdots \end{bmatrix}$$

$$(x_i - x_j)^T (u_i - u_j) = 0 \qquad \forall\, i,j$$

$$x^T \overbrace{\begin{bmatrix} I & I & \cdots & I \end{bmatrix}}^{c} \begin{array}{c} \\ e \end{array} \overbrace{\begin{bmatrix} (D\otimes I)_1 & 0 & & 0 \\ 0 & (D\otimes I)_2 & & \\ & & \ddots & \\ 0 & & & \end{bmatrix}}^{e} \overbrace{\begin{bmatrix} I & -I & & \\ & I & -I & \\ & & \vdots & \end{bmatrix} u}^{(D^T \otimes I)u}$$

$$D \otimes I = \begin{bmatrix} \begin{matrix} I \\ -I \end{matrix} & \begin{matrix} I \\ -I \end{matrix} & \cdots & \end{bmatrix}$$

$(D\otimes I)_e$

sum

prod.

in the expression $\underbrace{\sum_{(i,j)}}_{\text{sum}} \underbrace{(x_i - x_j)^T (u_i - u_j)}_{\text{prod.}}$

$$\begin{bmatrix} x^T & \cdots & x^T \end{bmatrix} \begin{array}{c} \\ e \end{array} \overbrace{\begin{bmatrix} (D\otimes I)_1 & 0 & & 0 \\ 0 & (D\otimes I)_2 & & \\ & & \ddots & \\ 0 & & & \end{bmatrix}}^{e} \overbrace{\begin{bmatrix} I & -I & & \\ & I & -I & \\ & & \vdots & \end{bmatrix} u}^{(D^T \otimes I)u} \qquad \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

arbitrary

sum

prod.

$R(G)$

overall translation

$z \in \mathbb{R}^p$

Prop:

if nullspace $\mathcal{R}(G) = \operatorname{span}\,(\mathbb{1} \otimes I_p)\, I$

then the structure is rigid.

nullspace of $(D^T \otimes I)$

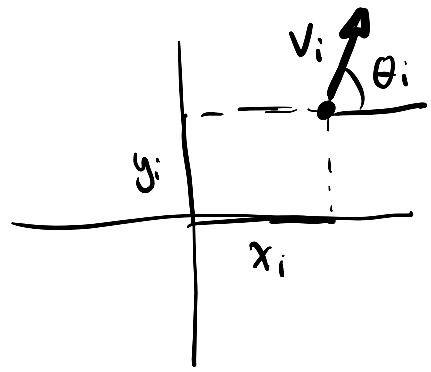
"the only motion that doesn't affect relative distances is an overall translation"

Control of Unicycle or Phase Dynamics

Agents: i

unicycle dynamics:

$$\begin{bmatrix} \dot{x}_i = v_i \cos\theta_i \\ \dot{y}_i = v_i \sin\theta_i \\ \dot{\theta}_i = \underbrace{\omega_i}_{\text{control}} \end{bmatrix}$$



state vector

$$\begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}$$

pos & orient.

control

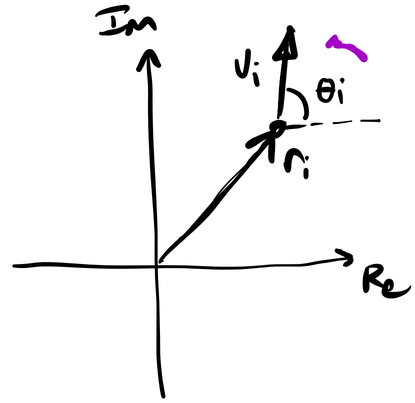
$$\underbrace{\begin{bmatrix} v_i \\ \omega_i \end{bmatrix}}_{\text{velocity}}$$

v_i : velocity

θ_i : heading angle

ω_i : change in heading angle

Represent with complex #'s



$$r_i(t) = x_i(t) + j\,y_i(t) \qquad j = \sqrt{-1}$$

$$\dot{r}_i = v_i e^{j\theta_i}$$

$$\dot{\theta}_i = \omega_i = u_i$$

normalize speed $v_i = 1$ "everyone moving w/ a constant speed"

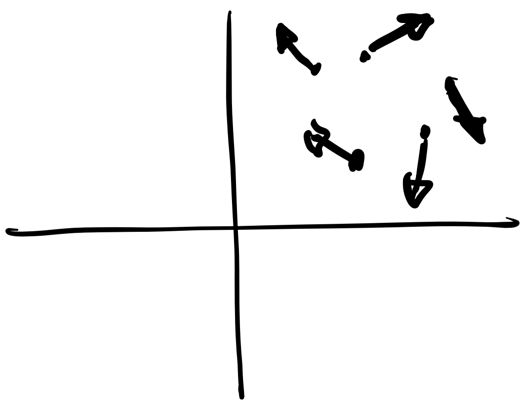
$$\theta = [\theta_1 \cdots \theta_n]^T$$

$$u = [u_1 \cdots u_n]^T$$

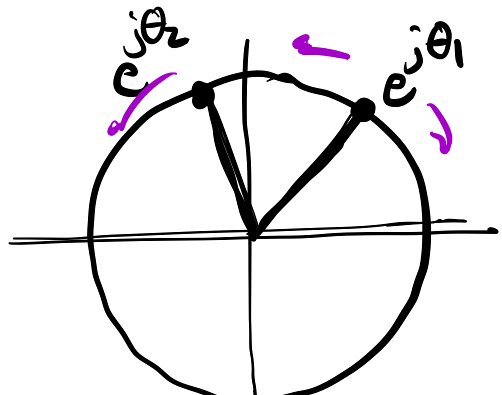
$$e^{j\theta} = [e^{j\theta_1} \cdots e^{j\theta_n}]^T$$

Dynamics of agents on the unit circle

"dynamics of phase angles"

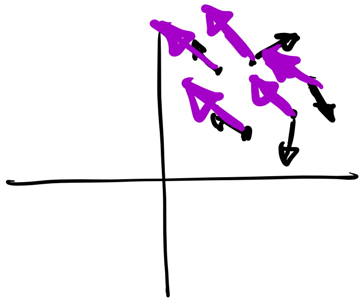


planar motion

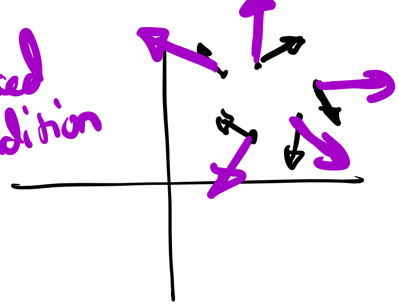


heading angle

Synchronization condition



Balanced condition



Potential function (Navigation function)

$\mathcal{U}(\theta) \rightarrow \dfrac{\partial \mathcal{U}}{\partial \theta_i}$ gives control for agent i

Define: "sum of exponentials"

$$p_m(\theta) = \frac{1}{nm} \mathbb{1}^T e^{jm\theta}$$

$$\mathcal{U}_m(\theta) = \frac{n}{2} \left| p_m(\theta) \right|^2 = \frac{1}{2nm^2} \left(e^{jm\theta}\right)^* \mathbb{1}\mathbb{1}^T e^{jm\theta}$$

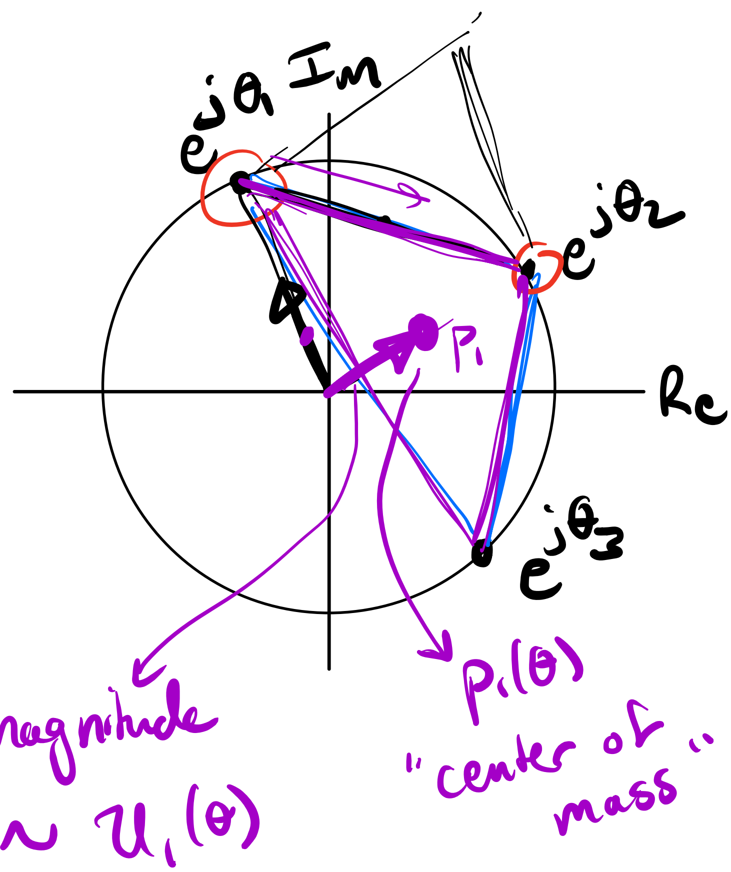
potential function of order m

just use when $m=1$ (for now.)

$$p_1(\theta) = \frac{1}{n} \mathbb{1}^T e^{j\theta}$$

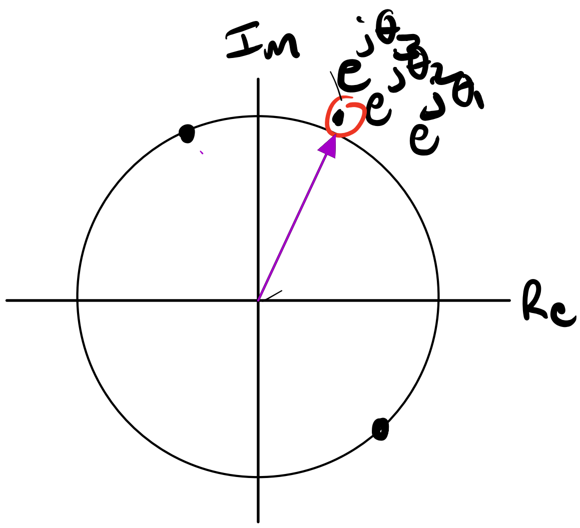
$$\begin{bmatrix} 1/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ e^{j\theta_3} \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

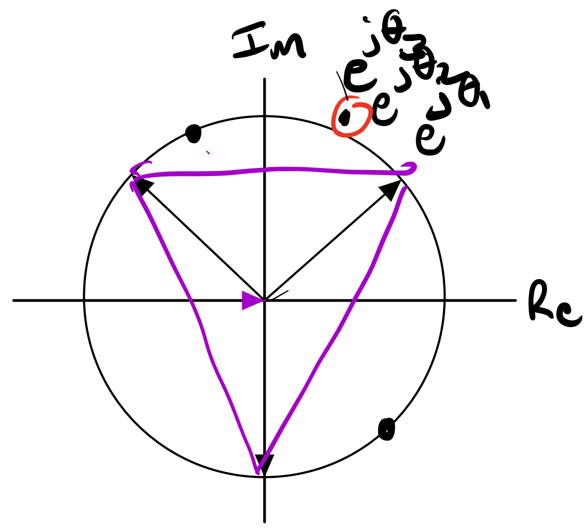


Maximize U_1 ...

Minimize U_1 ...



synchronization



balanced.

both correspond to critical points of potential

if use control $u_i = \dfrac{\partial U_1}{\partial \theta_i}$ $\rightarrow$ drive us to critical points

need to prove the sync & balanced cond are the only two stable critical points $\leftarrow$ can be done.

Prop.

unique max of $U_1(\theta)$ $\Rightarrow$ $\theta_i = \theta_j \mod (2\pi)$ sync

unique min of $U_1(\theta)$ $\Rightarrow$ $p_1(\theta) = 0$ balance

sign determines sync or balance

$\frac{\partial p_1}{\partial \theta_i}$

Control Law:

$$u_i(t) = -k \frac{\partial U_1}{\partial \theta_i} = -k \langle p_1(\theta), j e^{j\theta_i} \rangle$$

"chain rule"

$$= -\frac{k}{n} \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

complete graph

circular version of consensus

- for small $\theta_j - \theta_i$ $\rightarrow$ AP
- $\sin(\theta_j - \theta_i)$ never gets too

large / changes direction

for complete graph

$L = nI - \mathbb{1}\mathbb{1}^T \rightarrow$ in u_1