Discrete Consensus Dynamic:

Consider DAP dyn. on directed graph D: $\dot{x}(t) = -L(D)\,x(t)$

we first point out the difference between "discretization" versus " discrete sampling ".

Discretization:

- First, let's just naively discretize this dyn w/ stepsize Δ:

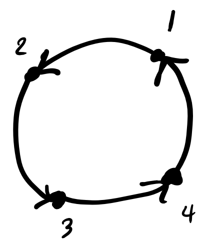
(Forward-Euler) $\quad \dfrac{x(t+1) - x(t)}{\Delta} = -L(D)\,x(t)$

or: (I) $\quad x(t+1) = [I - \Delta L(D)]\,x(t).$

note that this matrix respects the connection patterns of D

This may give us a discrete dynamic of consensus, But one has to be careful about the choice of the stepsize Δ.

Non-Example: Consider DAP dyn. on a directed cycle of 4 nodes.



Now, lets discretize this dyn w/ $\Delta = 1$.

Then: $x(t+1) = (I - \Delta L(D))\,x(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$

Do you see what does this discrete dynamic do ?!

→ So, in general, Δ has to be chosen sufficiently small and you can implement (I) in a distributed way!

Discrete sampling:

This time, let's just sample from the continuous DAP dyn. in δ-time intervals, i.e.:

$$\dot{x}(t) = -L(D)\,x(t)$$

sample $\rightarrow$ $z(k) = x(\delta k)$ for $k = 0, 1, 2, \ldots$.

Now, $z(k+1) = x(\delta(k+1)) = e^{-\delta(k+1)L(D)}\,x(0)$

because $\delta L(D)$, $\delta k L(D)$ commute! $\quad = e^{-\delta L(D)}\,e^{-\delta k L(D)}\,x(0) = e^{-\delta L(D)}\,x(\delta k)$

$$\Rightarrow \boxed{z(k+1) = e^{-\delta L(D)}\,z(k)}\,,\quad z(0) = x(0).$$

Now, the dyn. of $z(k)$ is the "true sampling" of DAP and we already know that under $\underbrace{\text{proper assumption}}_{\text{"}\exists \text{ a rooted out-branching in } D\text{"}}$ $z(k) \rightarrow \operatorname{span}\{\mathbb{1}\}$.

__Question:__ But, what is special about the matrix $e^{-\delta L(D)}$ that it can guarantee convergence to $\operatorname{span}\{\mathbb{1}\}$.

__Proposition:__ for all digraphs D and sampling intervals $\delta > 0$ $e^{-\delta L(D)}$ is a "stochastic matrix", i.e.

$$\underbrace{e^{-\delta L(D)}\,\mathbb{1} = \mathbb{1}}_{\text{row-sum}=1} \quad \text{and} \quad \underbrace{e^{-\delta L(D)} \geq 0}_{\text{non-negative matrix}}.$$

element_wise

Furthermore, the right/left eigenvectors of $e^{-sL(D)}$ are those of $L(D)$ associated w/ eigenvalues $e^{s\lambda_i}$, $i=1,2,\dots,n$.

Note: Don't confuse "non-negative matrices" with "positive-definite ones." It's just an unfortunate similarity in their names.

Proof:
$$e^{-sL(D)}\mathbf{1} = \left(\sum_{j=0}^{\infty} \frac{(-s)^j}{j!} L(D)^j\right)\mathbf{1} = \frac{(-s)^0}{0!} L(D)^0 \mathbf{1} = \mathbf{1}.$$

Notice that $\exists\, s$ (large enough) $\in \mathbb{R}_+ \ni -L(D)+sI \geq 0$.

Therefore, $-L(D)$ is a "Metzler" (or essentially non-negative) matrix.

Lemma: If C is metzler, then
$$e^{tC} \geq 0, \quad \forall\, t \geq 0$$

This now also suggest how to generalize both AP & DAP continuous dyn:

Suppose $\dot{x}(t) = A\,x(t)$ s.t.

A is metzler and $A\mathbf{1}=0$

digraph of A, $D=(\{i\}_1^n, E)$

$a_{k\ell}>0 \Leftrightarrow (\vec{v_\ell, v_k}) \in E$

Thm: [Moreau '04]

If D has a rooted out-branching

Then $x(t) \to \text{span}\{\mathbf{1}\}$.

More observations:

If D is balanced $\Rightarrow$ $1^T L(D) = 0$ $\Rightarrow$ $1^T e^{-sL(D)} = 1$

$\Rightarrow$ $e^{-sL(D)}$ has also column-sum $= 1$

$\Rightarrow$ $e^{-sL(D)}$ is a doubly stochastic matrix, i.e.,

a non-negative matrix with row-and-colum-sum zero.

Thm [Birkoff's]

Any doubly stochastic matrix is a convex combination of permutation matrices.

So, if D is balanced, then

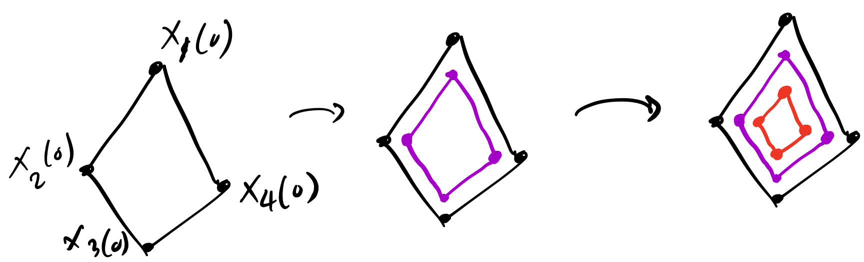
$$z(k+1) = e^{-sL(D)} z(k) = \sum_{i=1}^{k} \alpha_i P_i z(k)$$

$\uparrow$ permutation matrices

with $\alpha_i \geq 0$, $\sum \alpha_i = 1$.

$\Rightarrow$ "Every state of nodes, at any time in DAP, is a convex combination of the values of all nodes at the previous instance."

Picture:



Lemma: $\delta > 0$.

$$\left[e^{-\delta L(D)}\right]_{ij} > 0 \iff \begin{cases} i = j \\ \text{or} \\ \exists \text{ directed path from } j \to i. \end{cases}$$

Proof:

Notice: $e^{-\delta L(D)} = e^{-\delta\mu} \cdot e^{\delta(\mu I - L(D))}$. $\forall \mu > 0$.

$\Rightarrow$ zero-pattern in $e^{-L(D)}$ and $e^{\mu I - L(D)}$ are the same.

Chose μ large enough, say, $\mu^* > \max_i \{d_{in}(v_i)\}$.

$$\Rightarrow \begin{cases} L_+ = \mu^* I - L(D) \text{ is a non-negative matrix.} \\ \text{and} \\ [L_+]_{ij} > 0 \iff (\overrightarrow{j,i}) \in D. \end{cases}$$

Therefore, for any positive integer p:

$$\left[(L_+)^p\right]_{ij} > 0 \iff \exists \text{ a directed path of length } p \text{ from } j \to i \in D.$$

Finally, $e^{L_+} = \sum_{p=0}^{\infty} \frac{(L_+)^p}{p!}$, so

$$\left[e^{L_+}\right]_{ij} > 0 \iff \exists \text{ a directed path from } j \to i \in D.$$

$\square$

Corollary: D has a rooted out-branching if and only if (for any $\delta>0$) at least one of the columns of $e^{-\delta L(D)}$ is positive.

Now, suppose D is balanced and has a rooted out-branching.

Let us consider the following Lyapunov function:

$$V(z) = \max_i z_i - \min_i z_i$$

- Note that $V(z) \geq 0$ with equality if and only if $z \in \text{span}\{\mathbb{1}\}$.

- Consider $z(k+1) = e^{-\delta L(D)} z(k)$:

  - Recall that each state of $z(k+1)$ is a convex combination of the ones in $z(k)$.

  - Let $\left\{ \begin{array}{l} z_\ell(k) \text{ be the state of the node corresponding to the positive column of } e^{-\delta L(D)} \\ z_{\bar{i}}(k) = \max_i z_i(k), \quad z_{\underline{i}}(k) = \min_i z_i(k) \end{array} \right.$

$$z_{\bar{i}}(k+1) = \sum_{i \neq \ell} \alpha_i z_i(k) + \alpha_\ell z_\ell(k)$$

$$z_{\underline{i}}(k+1) = \sum_{i \neq \ell} \beta_i z_i(k) + \beta_\ell z_\ell(k)$$

$$z_{\bar{i}}(k+1) - z_{\underline{i}}(k+1) \leq z_{\bar{i}}(k)(1-\alpha_\ell) + \alpha_\ell z_\ell(k) - \left((1-\beta_\ell) z_{\underline{i}}(k) + \beta_\ell z_\ell(k) \right)$$

$$= z_{\bar{i}}(k) - z_{\underline{i}}(k) + \alpha_\ell \left(z_\ell(k) - z_{\bar{i}}(k) \right) + \beta_\ell \left(z_{\underline{i}}(k) - z_\ell(k) \right)$$

≤ 0 with equality iff $z_{\bar{i}}(k) = z_\ell(k) = z_{\underline{i}}(k)$.

see the paper by Moreau for the generalization of DAP dyn. using the following dynamics:

$$V(x) = \tfrac{1}{2} x^T x \quad \text{and} \quad V(x) = \max_{i \in |D|} x_i - \min_{i \in |D|} x_i .$$