Review:

- The DAP $\dot{x} = -L(D)x$ reaches average consensus from every initial condition if and only if $D$ is weakly connected and balanced.

- Cartesian product of (undirected) graphs $G_1, G_2$ denoted by $G = G_1 \square G_2$.
- properties of Cartesian products
- [Prime factorization of graphs]:
  Every connected graph has a unique prime factorization w.r.t. Cartesian product.

- $L(G_1 \square G_2) = L(G_1) \otimes I_m + I_n \otimes L(G_2)$

  Kronecker product of matrices.

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Next, we try to understand the spectrum of $L(G_1 \square G_2)$ w.r.t. the ones for $L(G_1)$, $L(G_2)$.

Lemma: assume $\begin{cases} \lambda_1, \dots, \lambda_n & \text{eigenvalues of } L(G_1) \\ \mu_1, \dots, \mu_m & \text{eigenvalues of } L(G_2) \end{cases}$

associated with $\begin{cases} u_1, \dots, u_n & \text{eigenvectors of } L(G_1) \\ v_1, \dots, v_m & \text{" } \quad \text{" } L(G_2) \end{cases}$.

kronecker

Then, $u_i \otimes v_j$ is the eigen vector of

$L(G_1 \square G_2)$ associated w/ the eigenvalue $\lambda_i + \mu_j$,

for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Proof:

$$\begin{aligned} L(G_1 \square G_2)(u_i \otimes v_j) &= (L(G_1) \otimes I_m)(u_i \otimes v_j) + (I_n \otimes L(G_2))(u_i \otimes v_j) \\ &= (L(G_1) u_i) \otimes v_j + u_i \otimes (L(G_2) v_j) \\ &= \lambda_i u_i \otimes v_j + \mu_j u_i \otimes v_j \\ &= (\lambda_i + \mu_j) u_i \otimes v_j \end{aligned}$$

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Thm [Factorization lemma for AP on $G$]:

suppose $G = G_1 \square G_2 \square \cdots \square G_n$ and

$$\dot{x}_i = -L(G_i) x_i(t) \quad w/ \quad x_i(0) = \begin{bmatrix} x_{i,1}(0) \\ x_{i,2}(0) \\ \vdots \\ x_{i,|V_i|}(0) \end{bmatrix}$$

for $i = 1, \ldots, n$.

Then, the AP on $G$ (i.e. $\dot{x}(t) = -L(G) x(t)$) follows

$$x(t) = x_1(t) \otimes x_2(t) \otimes \cdots \otimes x_n(t)$$

w/ initial condition $x_1(0) \otimes x_2(0) \otimes \cdots \otimes x_n(0)$.

Proof: Note that

$$G = G_1 \square G_2 \square \cdots \square G_n = \Big( \cdots \big( (G_1 \square G_2) \square G_3 \big) \square \cdots \Big) \square G_n$$

Therefore, it suffices to show this for $n=2$. Thus, supp. $G = G_1 \square G_2$ and recall

$$L(G) = L(G_1) \otimes I_\ell + I_k \otimes L(G_2) \qquad \begin{cases} k = |G_1| \\ \ell = |G_2| \end{cases}$$

Now, let for $i = 1, 2$:

$$\dot{x}_i(t) = -L(G_i)\, x_i(t) \quad \text{with } x_i(0) \text{ given.}$$

and define $x(t) \triangleq x_1(t) \otimes x_2(x)$. Then

$$\begin{aligned}
\dot{x}(t) &\overset{\text{(why?)}}{=} \dot{x}_1(t) \otimes x_2(t) + x_1(t) \otimes \dot{x}_2(t) \\
&= \big(-L(G_1)\, x_1(t)\big) \otimes x_2(t) + x_1(t) \otimes \big(-L(G_2)\, x_2(t)\big) \\
&= -\big(L(G_1)\, x_1(t)\big) \otimes \big(I_\ell\, x_2(t)\big) - \big(I_k\, x_1(t)\big) \otimes \big(L(G_2)\, x_2(t)\big) \\
\text{why?!} \quad &= -\big(L(G_1) \otimes I_\ell\big)\big(x_1(t) \otimes x_2(t)\big) - \big(I_k \otimes L(G_2)\big)\big(x_1(t) \otimes x_2(t)\big) \\
&= -L(G)\, x(t).
\end{aligned}$$

∎

Question: Under what condition does $x(t)$ converges?!
What is the rate of convergence?!

## New Approach:

Next, we would like to ask more complicated questions that requires different techniques; e.g.

"what happens in (AP) if the underlying graph $\mathcal{G}$ or $\mathcal{D}$ is changing during the evolution of states?"

we use Lyapunov techniques and its generalizations to answer these kind of questions!

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Lyapunov theory: (see Appendix 3 in [Meshahi '10])

Suppose $\quad \dot{x} = f(x(t))$, $x(0)$=given $\quad$ s.t. $f(0)=0$.

Def: we say origin is "stable" if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ \ni \ \left( \|x(0)\| \le \delta \Rightarrow \|x(t)\| \le \varepsilon, \ \forall t \ge 0 \right)$$

we say origin is "asymptotically stable" (AS) if

origin is stable and $\exists \delta > 0 \ \ni \left( \|x(0)\| \le \delta \Rightarrow x(t) \to 0 \text{ as } t \to \infty \right)$

we say origin is "globally asymptotically stable" (GAS) if

origin is (AS) for arbitrary $x(0)$.

**Thm:** If there exist a "Lyapunov function" $V: \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\begin{cases} V(0) = 0 \\ V(x) \geq 0 \quad \text{with equality iff } x = 0 \\ \frac{d}{dt}\left(V(x(t))\right) < 0 \quad \text{whenever } x(t) \neq 0. \end{cases}$$

then the origin is asymptotically stable. In addition, if $V(x) \to \infty$ as $\|x\| \to \infty$, the origin is (GAS).

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Let's see if we can use this for (AP):

$\dot{x} = -L(G)x$, define: $V(x(t)) := \frac{1}{2} x^T(t)\, x(t) = \frac{1}{2}\|x(t)\|^2$

then $\dot{V}(t) = \frac{d}{dt}\left(V(x(t))\right) = x^T(t)\,\dot{x}(t) = -x^T(t)\, L(G)\, x(t)$

$L(G)$ is P.S.D $\Longrightarrow$ $\dot{V}(t) \leq 0$;

but it is not strictly $< 0$; (recall that $L(G)\mathbb{1} = 0$)

Here, $V(t)$ is NOT a Lyapunov function;

instead, we call it a "weak Lyapunov function".

Question: what can we guarantee for a system w/ a weak Lyapunov function?

Thm: [La Salle's Invariance Principle]

$$\dot{x} = f(x(t)), \quad x(0) = \text{given}, \quad f(0) = 0.$$

V: weak Lyapunov func s.t. $V(x) \to \infty$ as $\|x\| \to \infty$.

M: largest invariant set contained in $\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$.

Then,
$$\inf_{y \in M} \|x(t) - y\| \to 0 \quad \text{as } t \to \infty.$$

---

Back to our AP dynamics

if G is connected.

$$\{x \in \mathbb{R}^n \mid \dot{V}(t) = 0\} = \{x \in \mathbb{R}^n \mid x^T L(G) x = 0\} = \text{span}\{1\}.$$

and as $\dot{x}(t) = 0$ if $x(t) \in \text{span}\{1\} \Rightarrow M = \text{span}\{1\}$

Thus, by La Salle's Invariance Principle,

$$x(t) \longrightarrow \text{span}\{1\}.$$

---

what about the DAP dynamics?

$\dot{x}_{(t)} = -L(D)\, x(t)$ define $V(x(t)) = \frac{1}{2} x(t)^T x(t)$

not symm.

$$\Rightarrow \dot{V}(t) = x^T(t)\, \dot{x}(t) = -x^T(t)\, L_{in}(D)\, x(t)$$

By Gersgorian disk thm $\leq 0$

not strictly $< 0$. $\Rightarrow$ weak Lyapunov funct.

If $D$ is strongly connected then, the largest invariant set in

$$\{x \in \mathbb{R}^n \mid \dot{V}(t)=0\} = \{x \mid x^T(L(D)+L(D)^T)x=0\}$$

is the null space of $L(D)$ which is $\text{span}\{\mathbf{1}\}$. (why?)

$\Rightarrow$ By, LaSalle's Inv. Prin., $x(t) \longrightarrow \text{span}\{\mathbf{1}\}$.

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what if $D$ is not strongly connected, yet contains a rooted out-branching? $\Rightarrow$ redefine $V(z) = \max_i z_i - \min_j z_j$.

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## Switched Agreement Protocol:

Consider finitely many strongly connected digraphs

$$\{D_1, \dots, D_k\}.$$

switched AP

Suppose $\dot{x}(t) = -L(D_i)x(t)$ with $i \in \{1, \dots, k\}$.

This is a "switched linear system" and described by

"Differential Inclusion" $\dot{x}(t) \in \{-L(D_i)x(t) \mid i \in \{1, \dots, k\}\}$.

Considering $V(x(t)) = \frac{1}{2}x^T(t)x(t)$, we get

$$\dot{V}(t) \in \{-x^T(t)L(D_i)x(t) \mid i \in \{1, \dots, k\}\}.$$

where each dynamic vanishes on:

$$F_i = \{ x \in \mathbb{R}^n \mid x^T ( L(D_i) + L(D_i)^T ) x \}.$$

But, as each $D_i$ is strongly connected,

$$F_i = \text{span}\{\mathbb{1}\} \quad \text{for every } i \in \{1, \ldots, k\}$$

we call $V(t)$ here a "Common weak Lyapunov function"

for the switched agreement protocol.

$\Rightarrow$ A generalization of LaSalle's Inv. principle [Thm A.9 in Meshahi '10]

still implies that $\quad x(t) \longrightarrow \text{span}\{\mathbb{1}\}$.

<u>Thm A.9</u>: Suppose $V$ is a common weak Lyapunov function

for the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \quad , \sigma(t) \in S = \{1, \ldots, k\}$$

↖ switching mechanism.

Let $M_i$ be the largest invariant set under mode $i$

that is contained in

$$\left\{ x \in \mathbb{R}^n \,\middle|\, \left[\frac{\partial V(x)}{\partial x}\right]^T f_i(x) = 0 \right\}.$$

If $M_i = M_j = M^*$ for all $i, j \in S$, then $x(t) \longrightarrow M^*$ as $t \to \infty$.