Kalman Filter (Discrete Time)

Major sources:

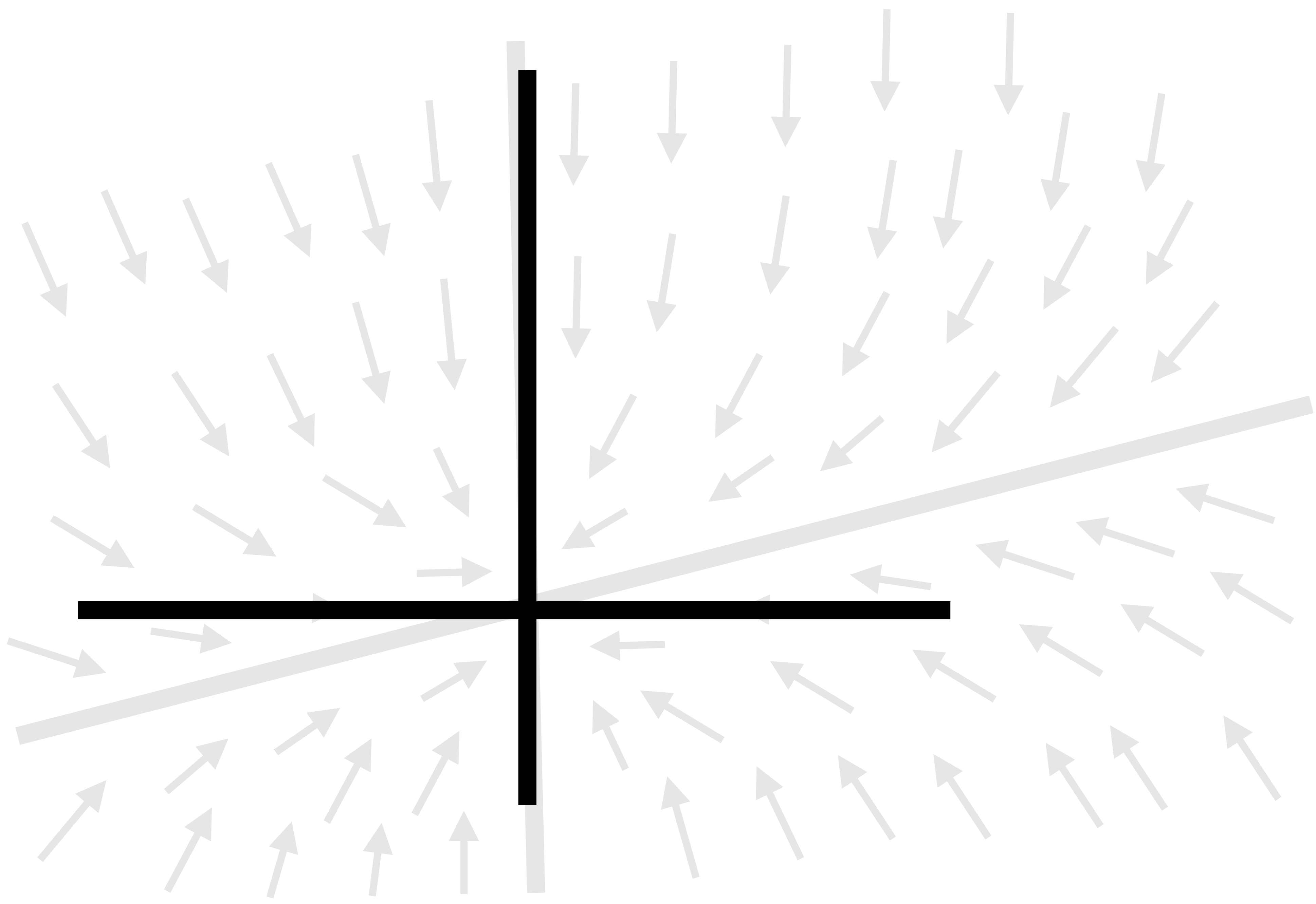
Spring 2022 - Dan Calderone

Discrete Time Kalman Filter

Dynamics: $x(k+1) = Ax(k) + w(k)$ $\quad w(k) \sim \mathcal{N}(0, W)$

State-Space

$x \in \mathbb{R}^2$



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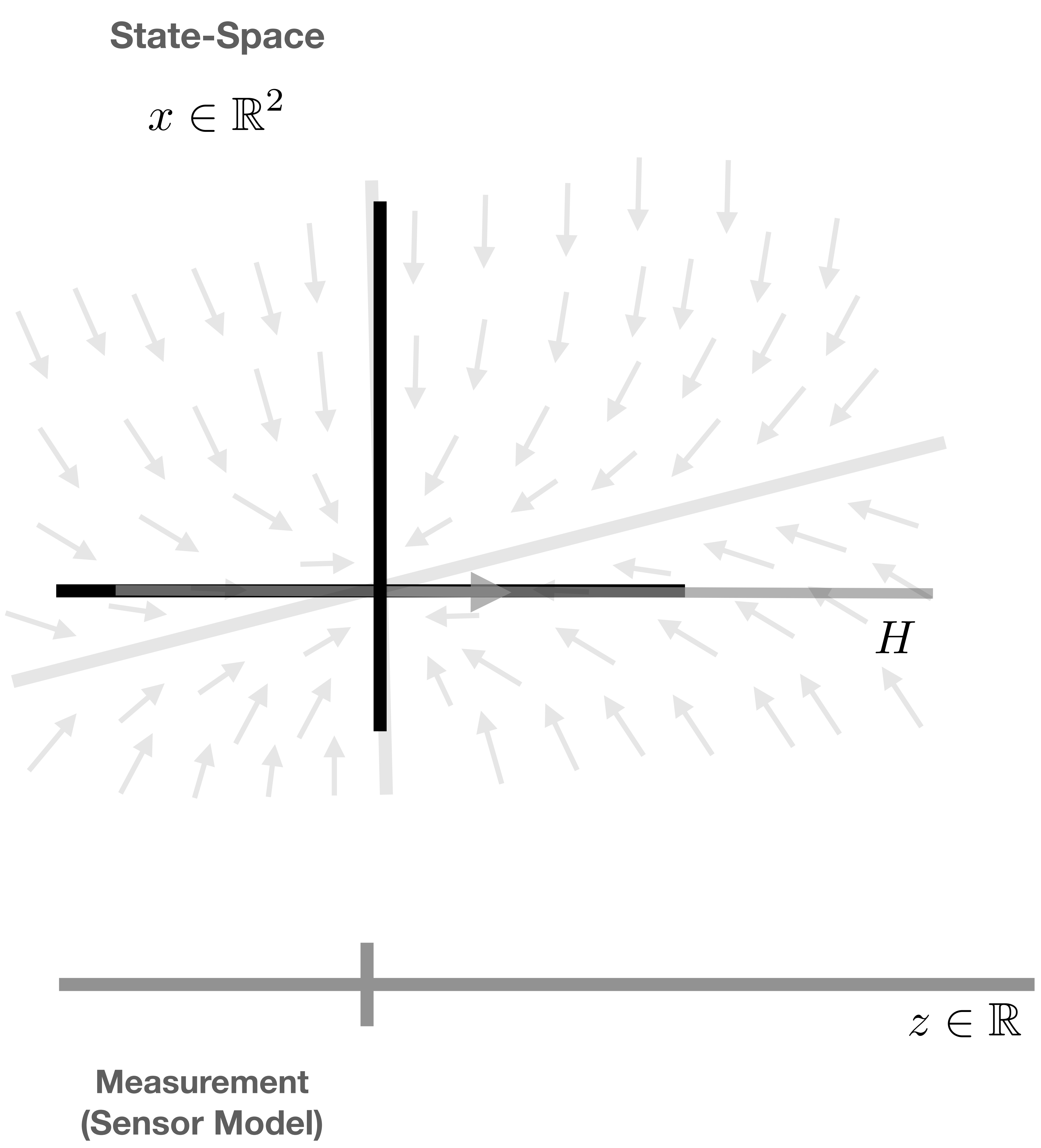

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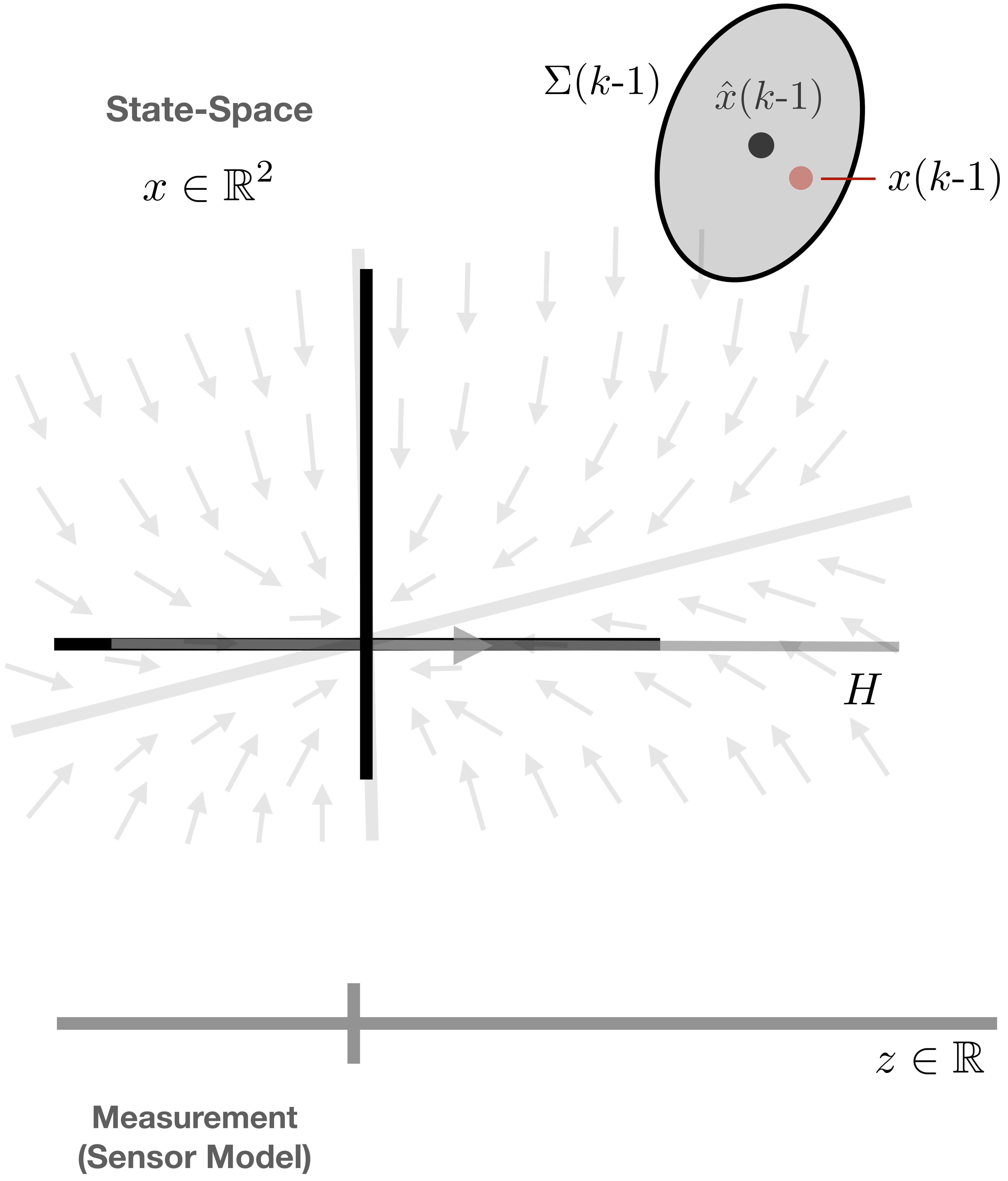

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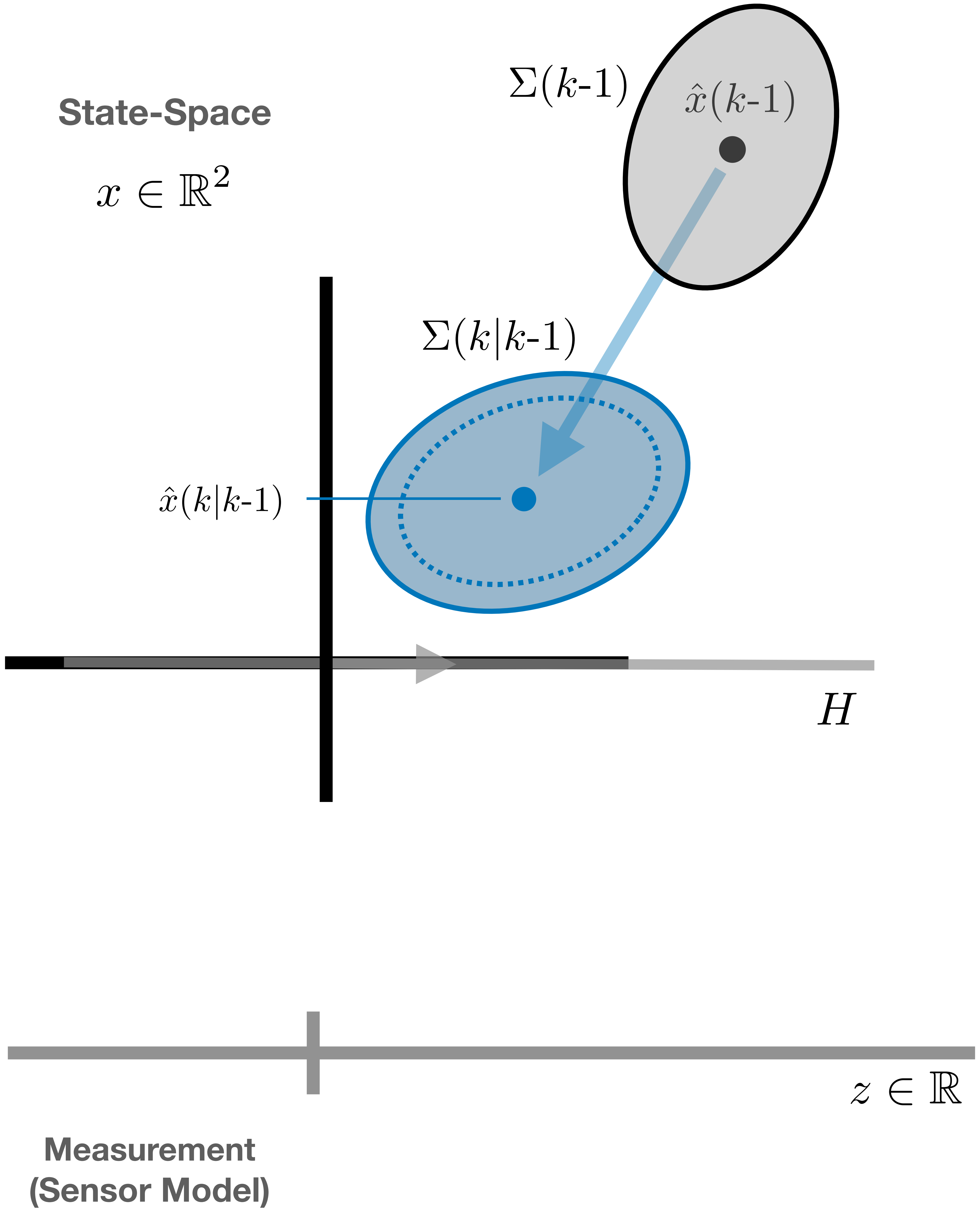

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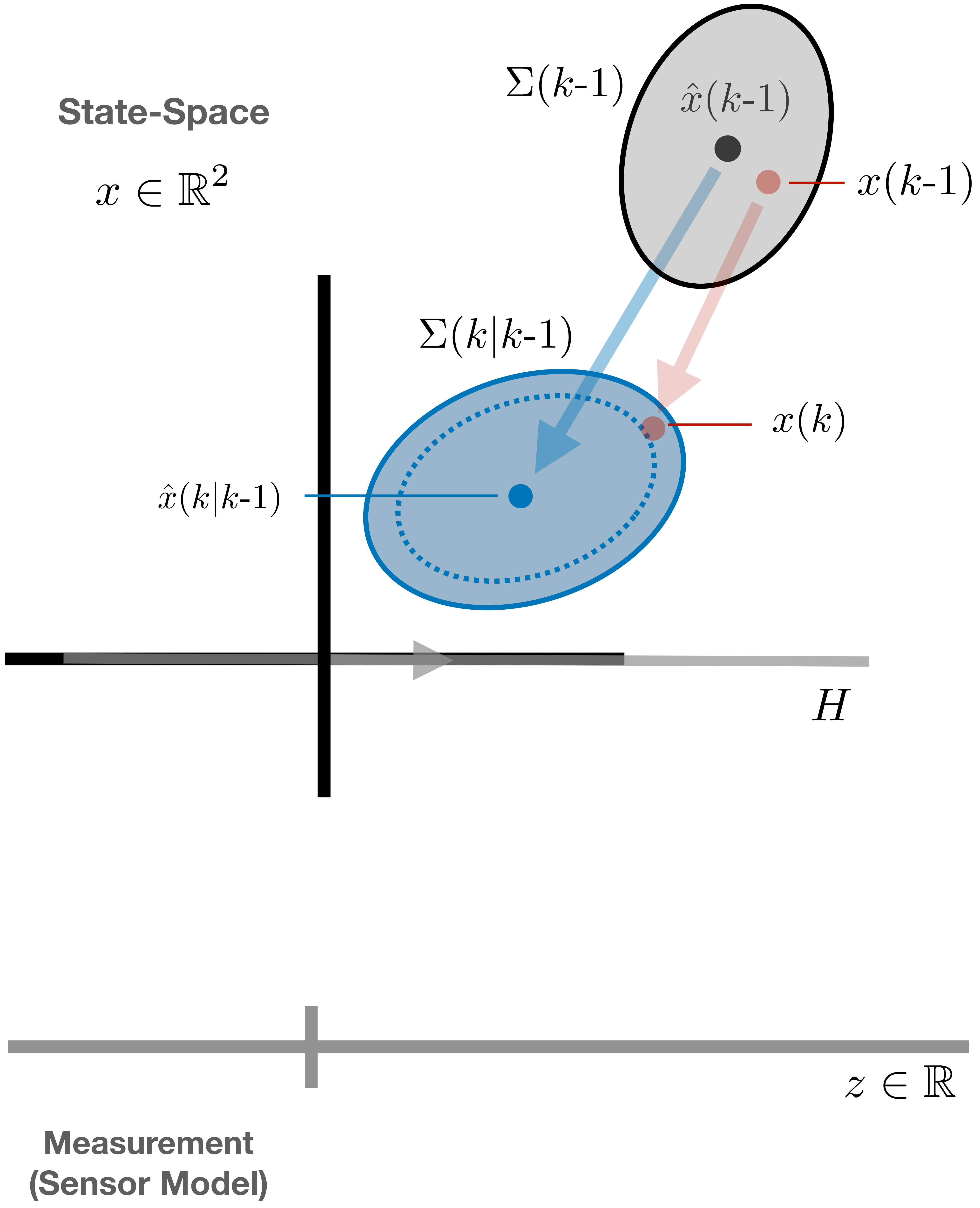

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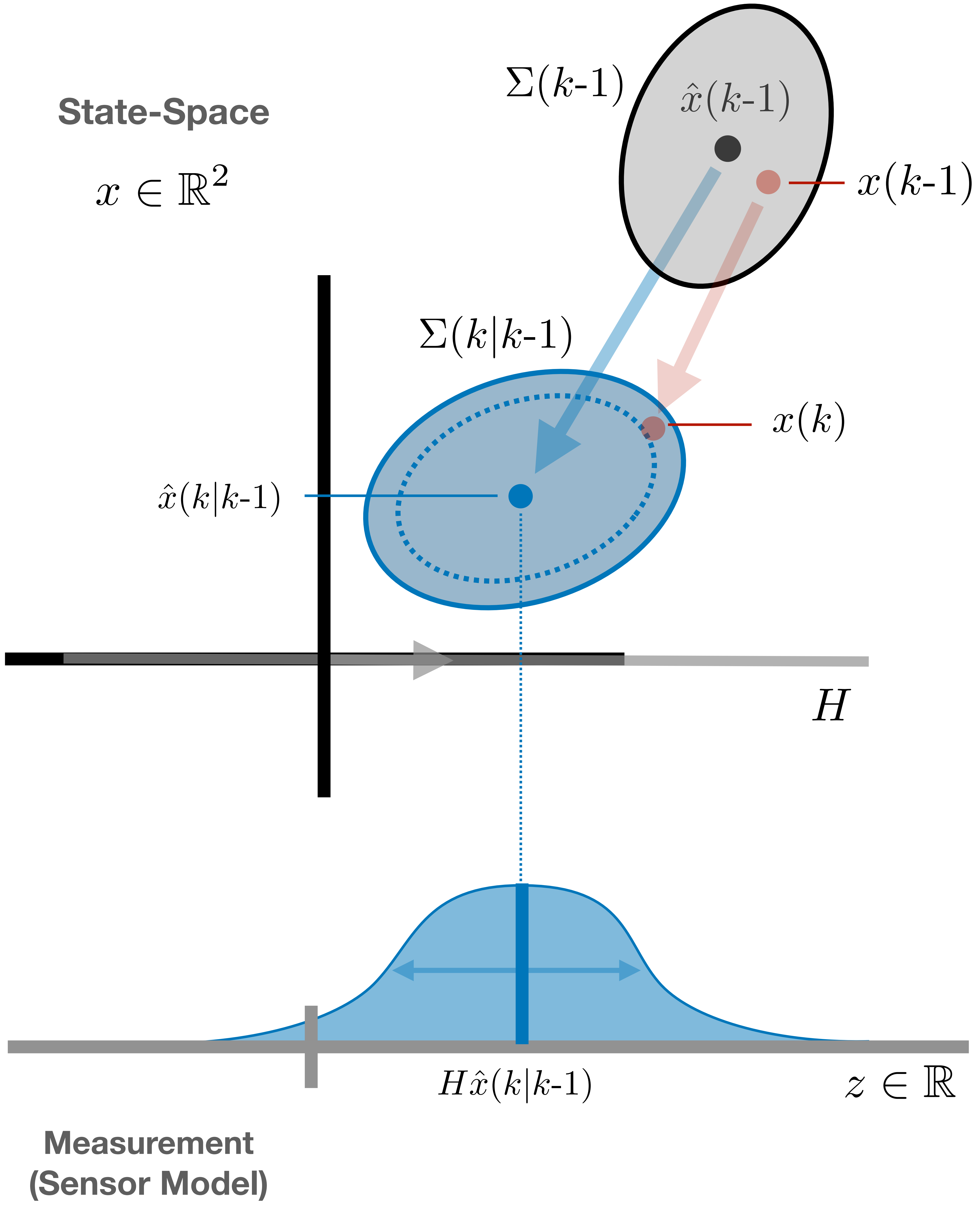

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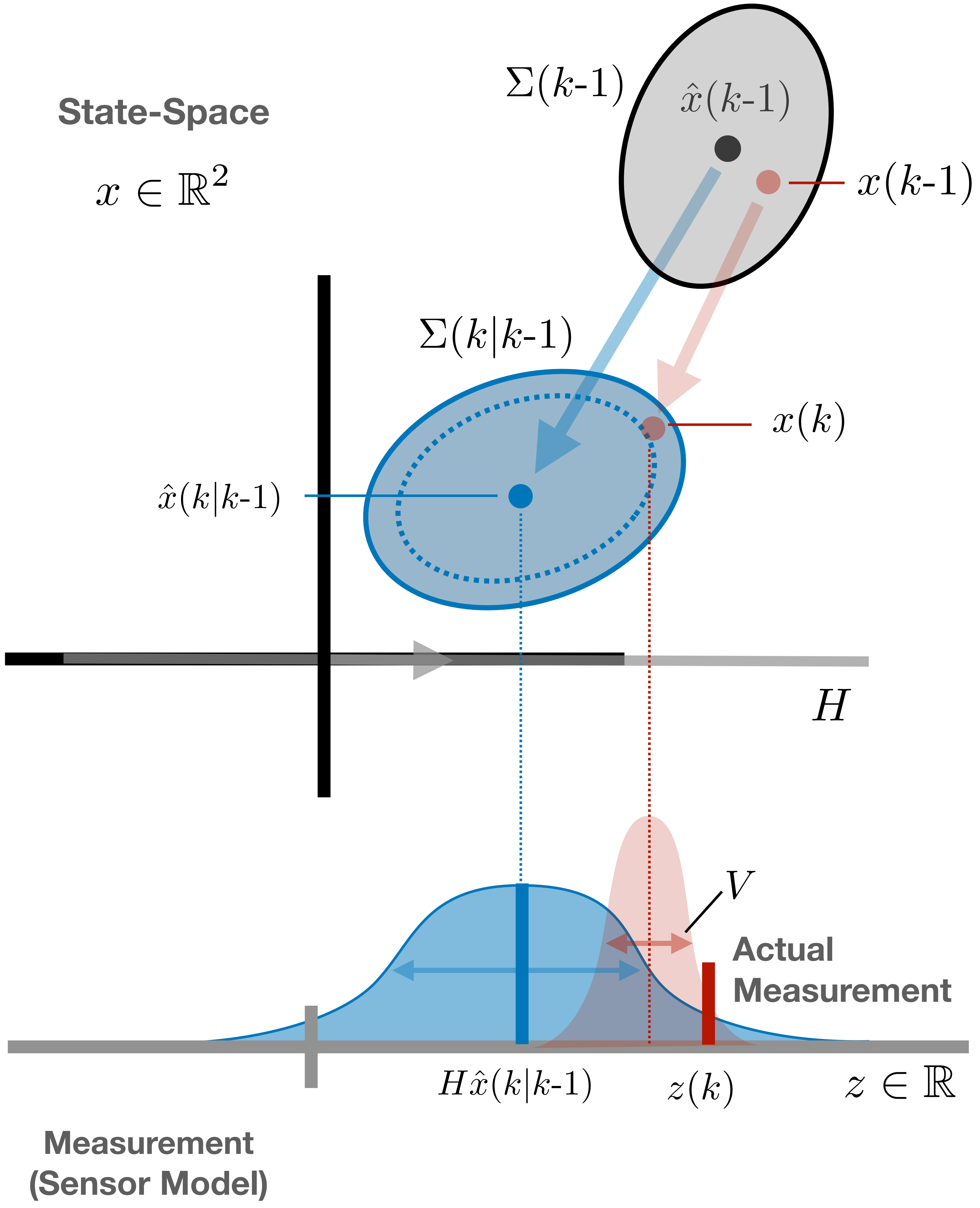

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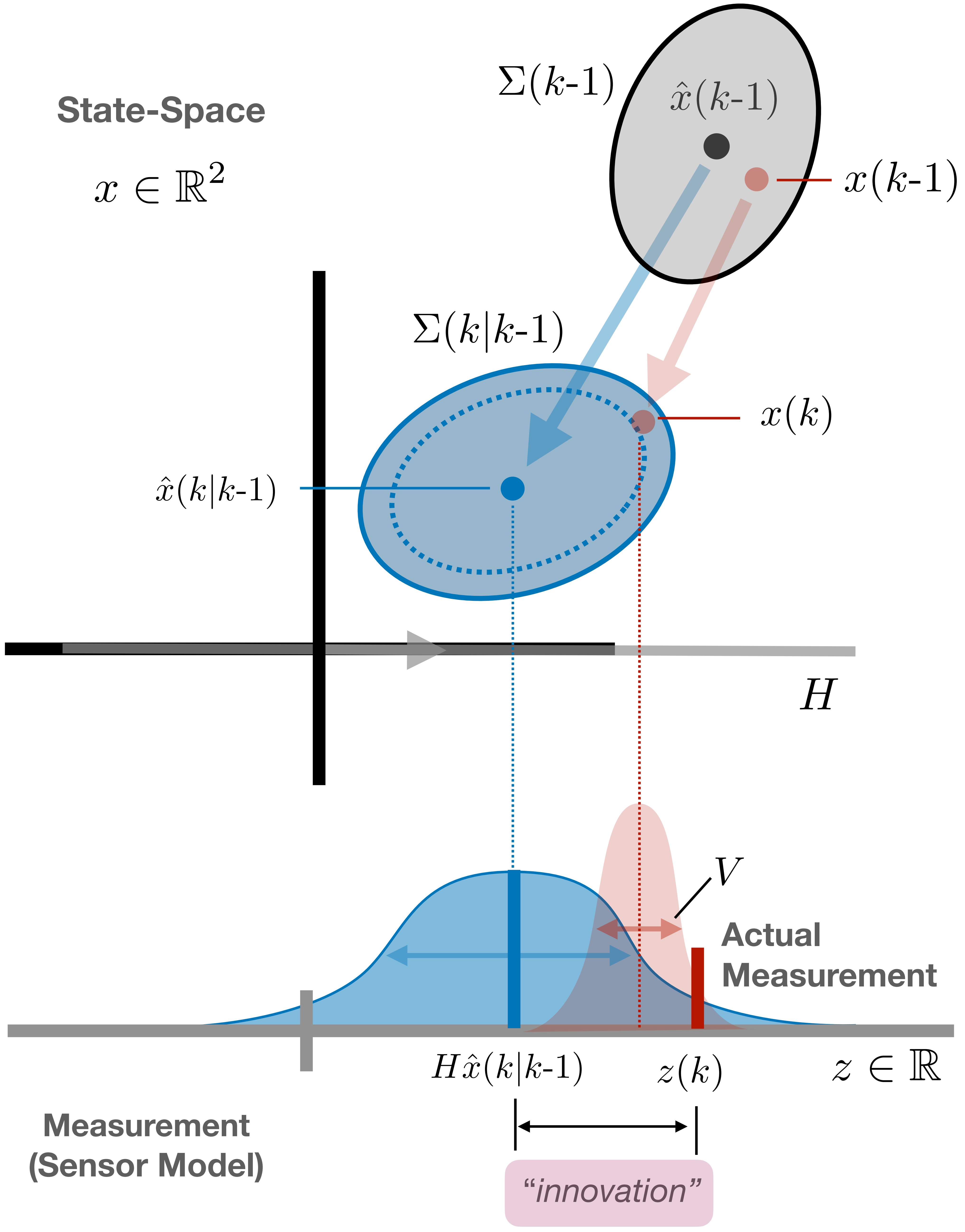

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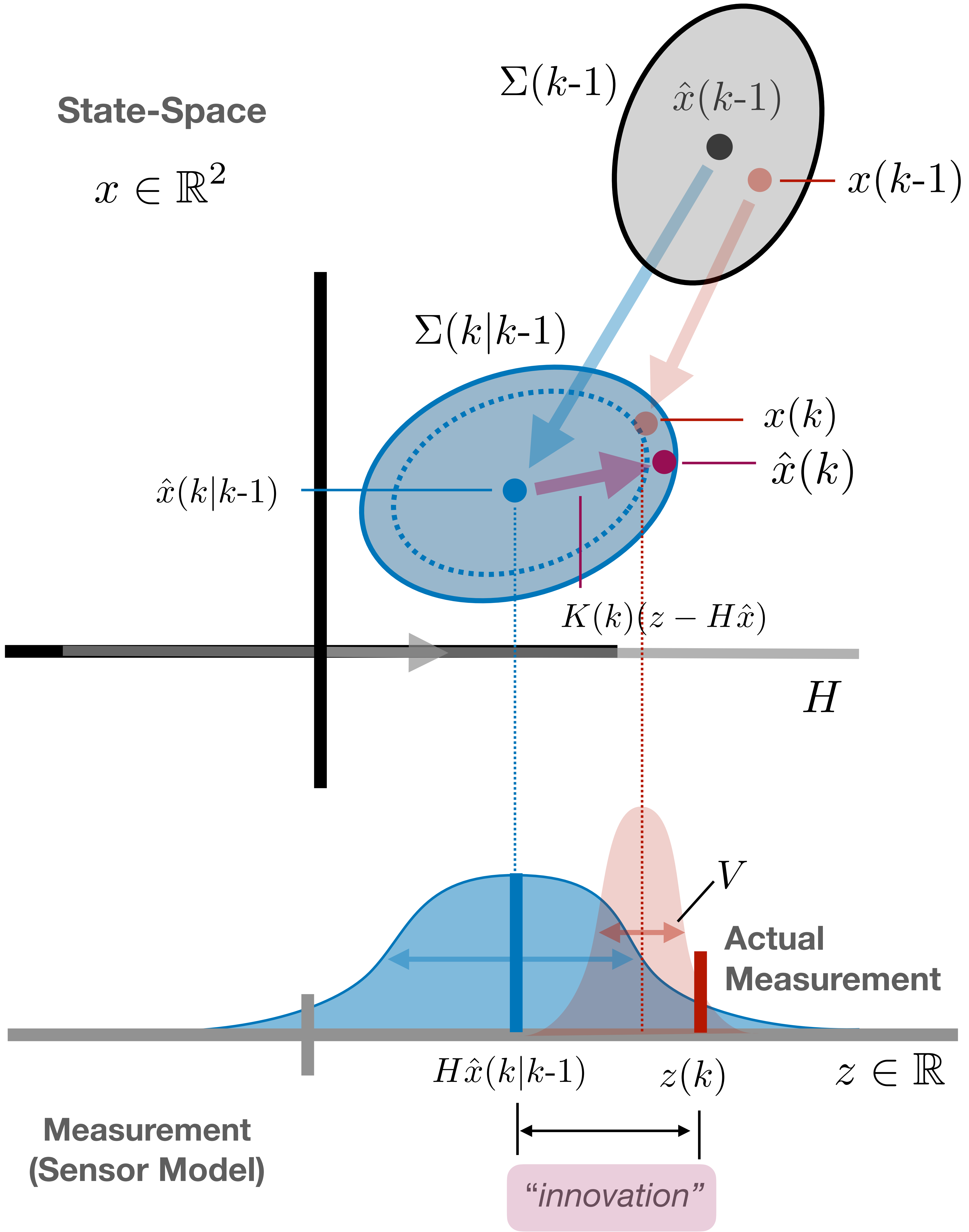

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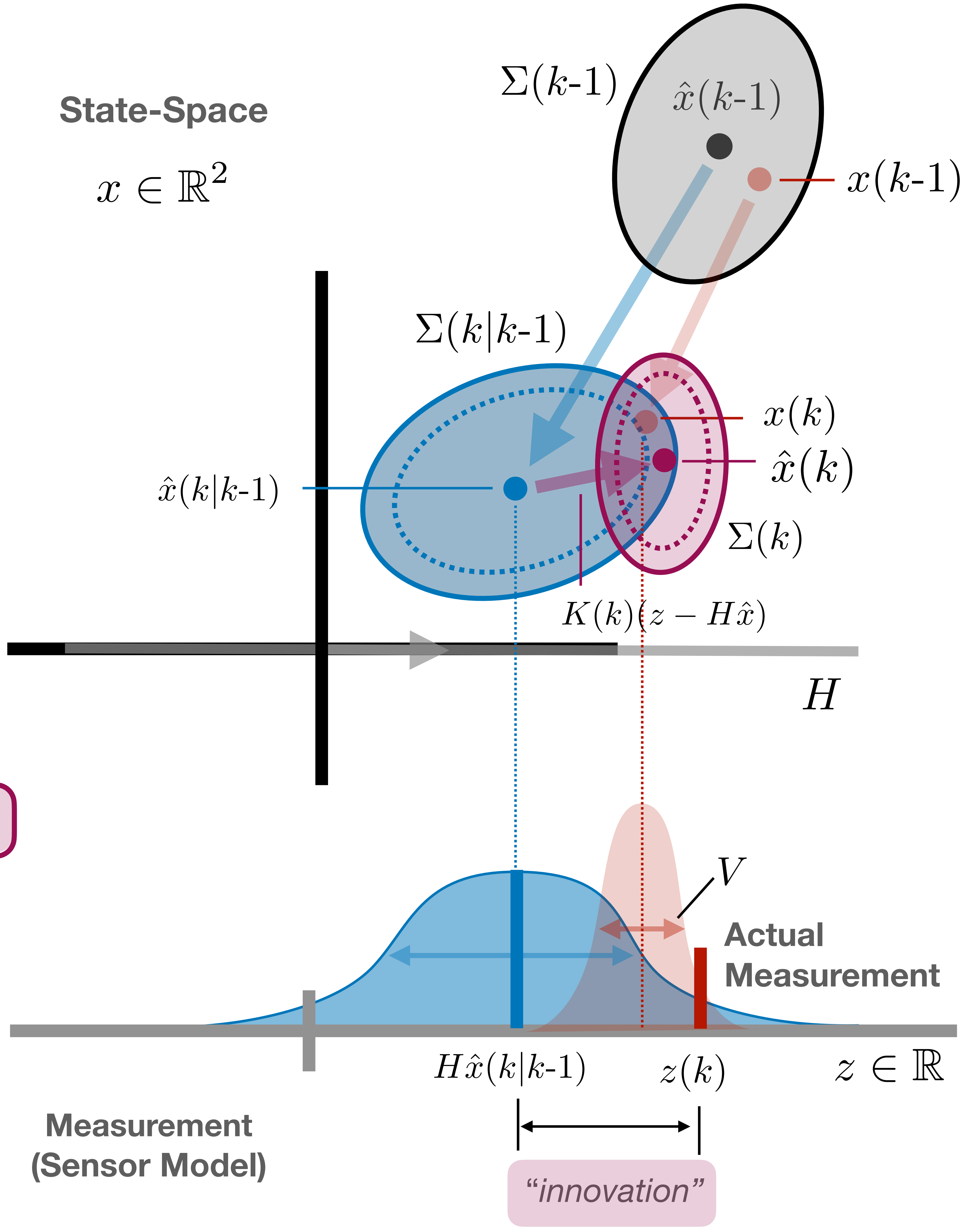

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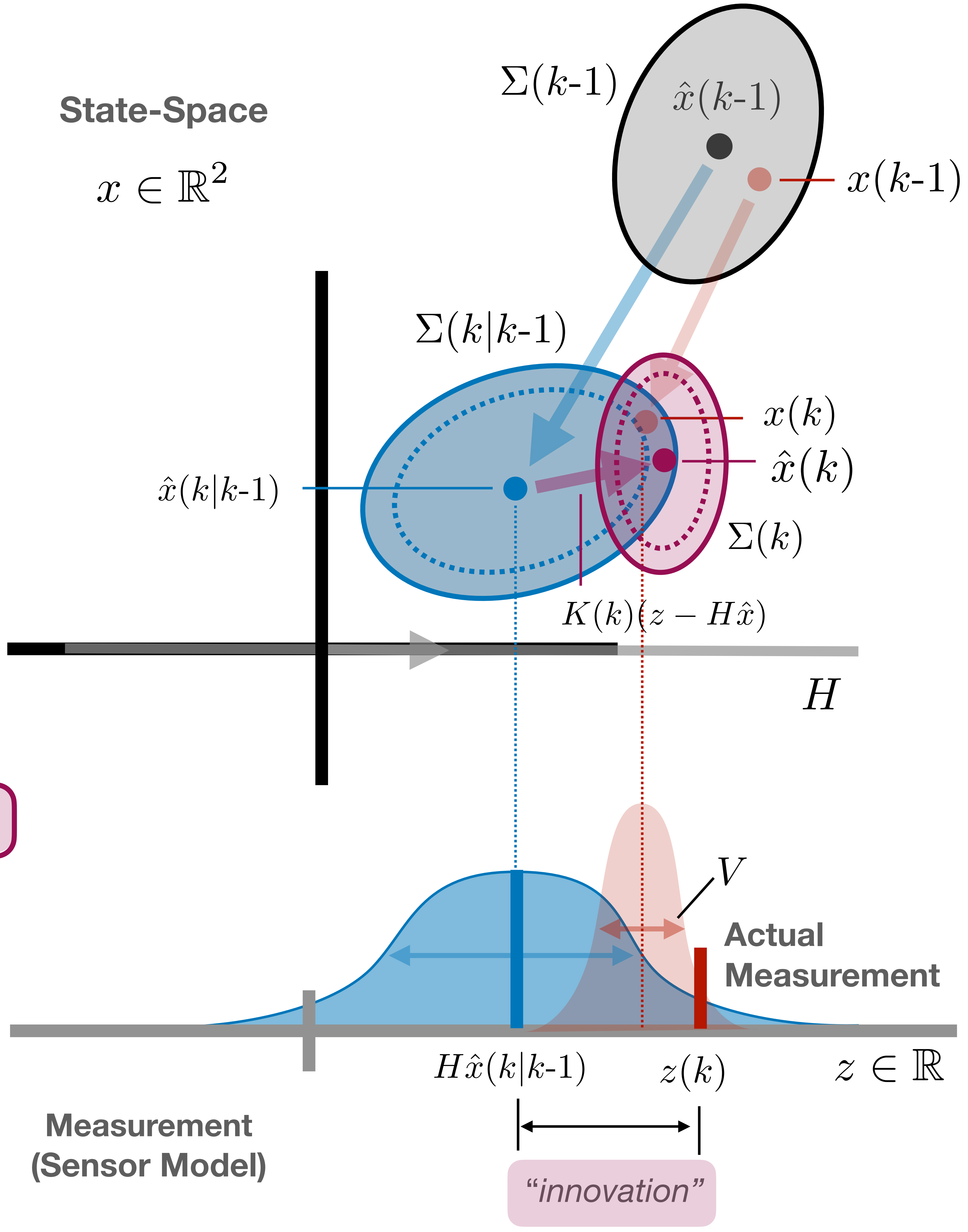

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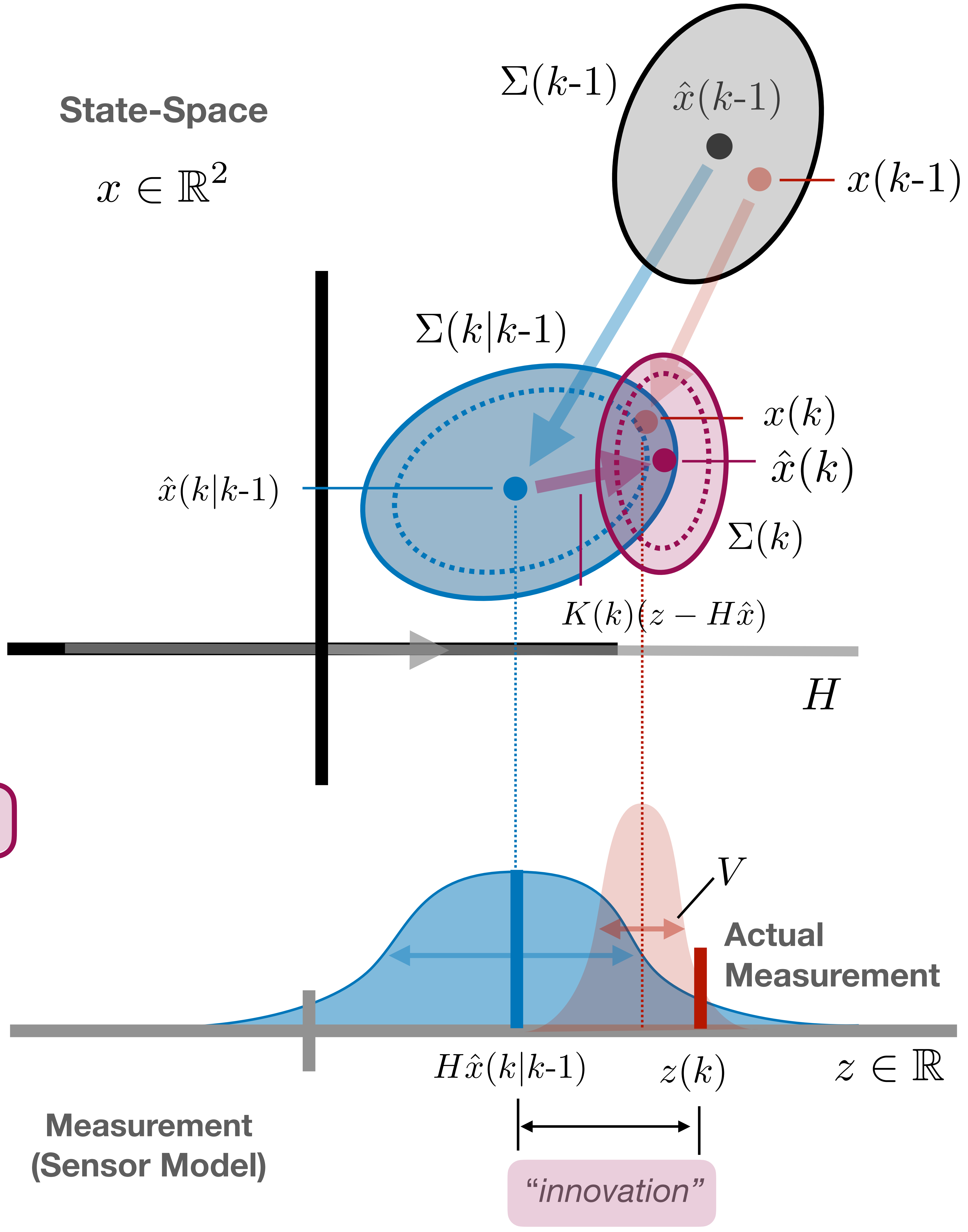

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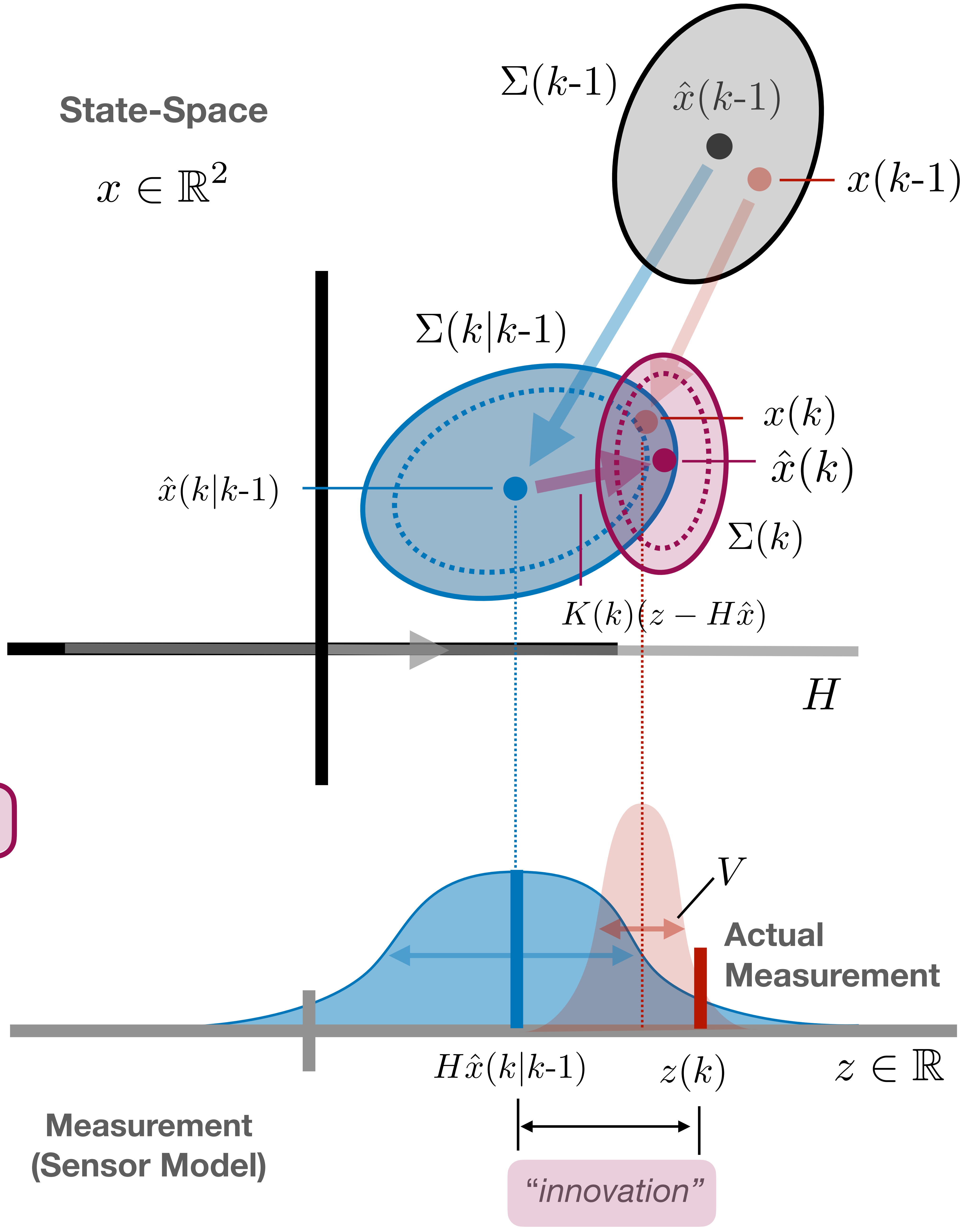

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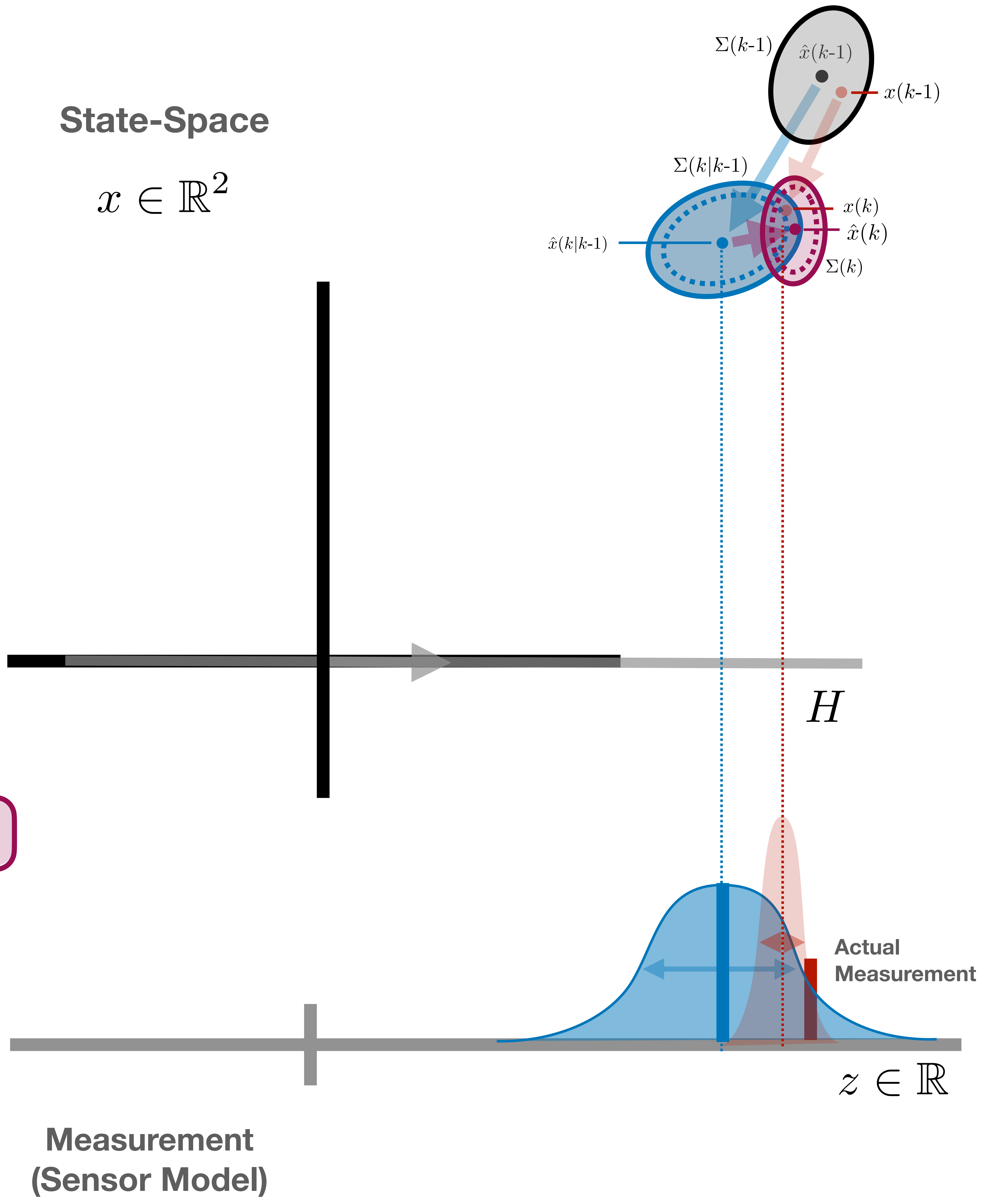

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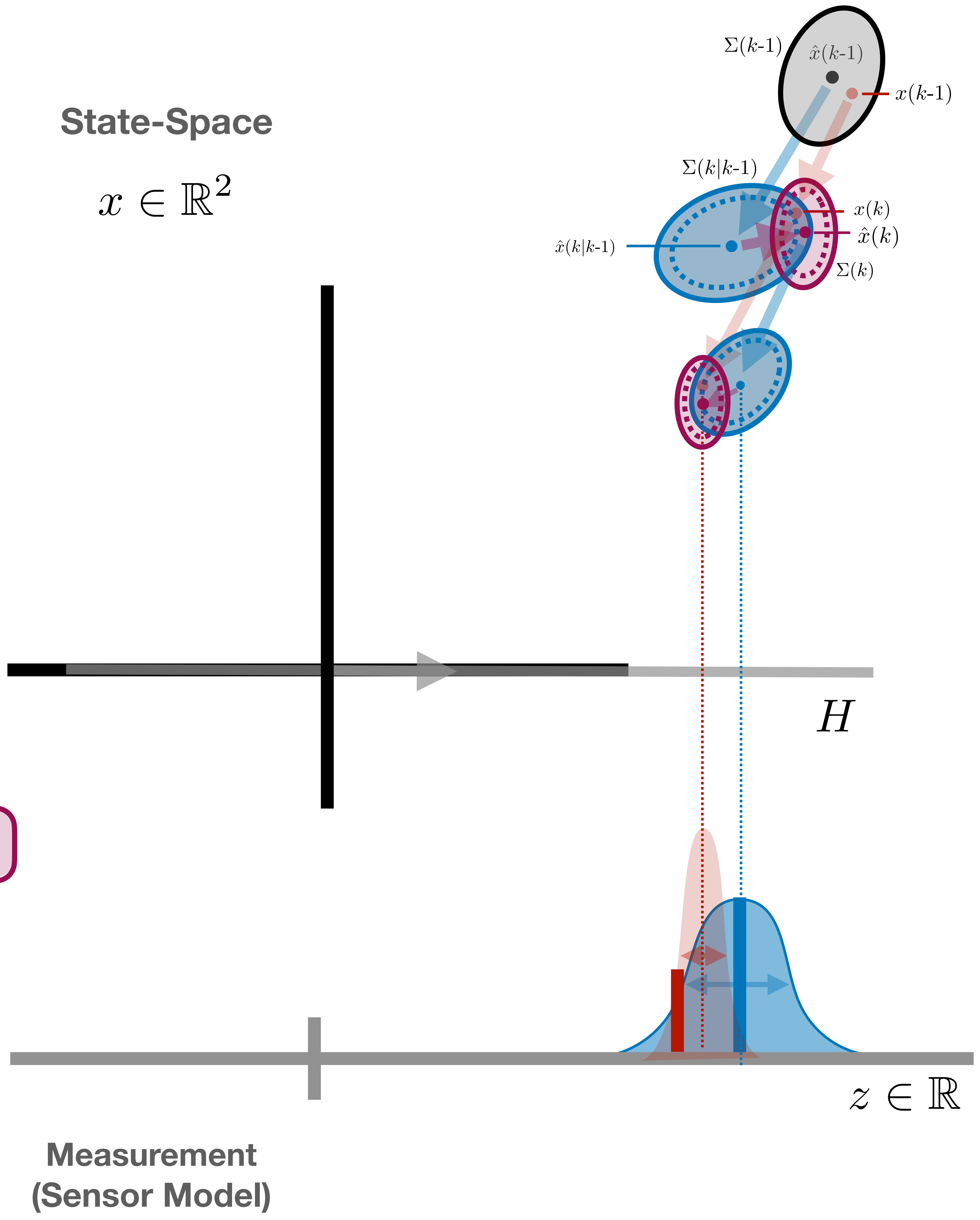

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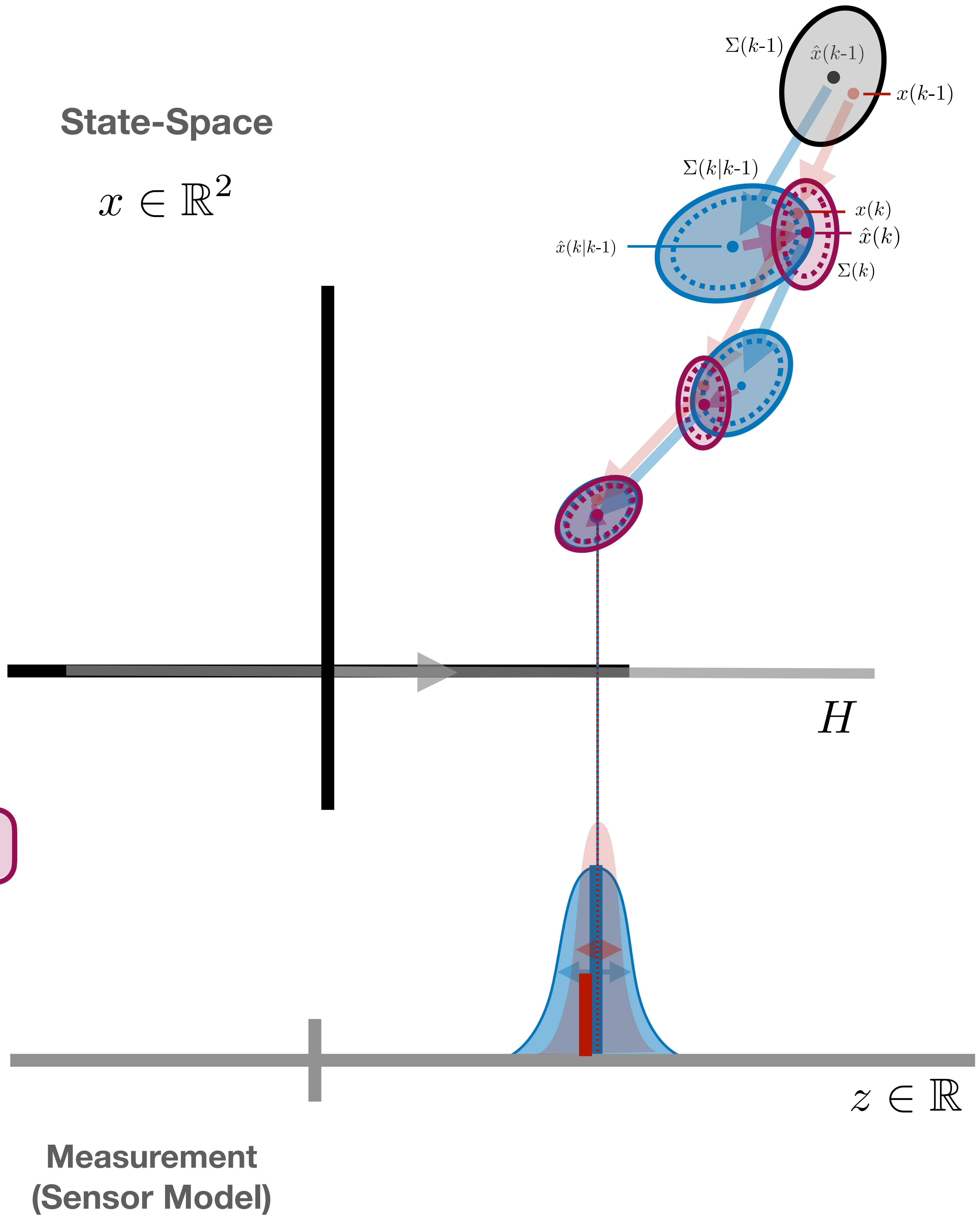

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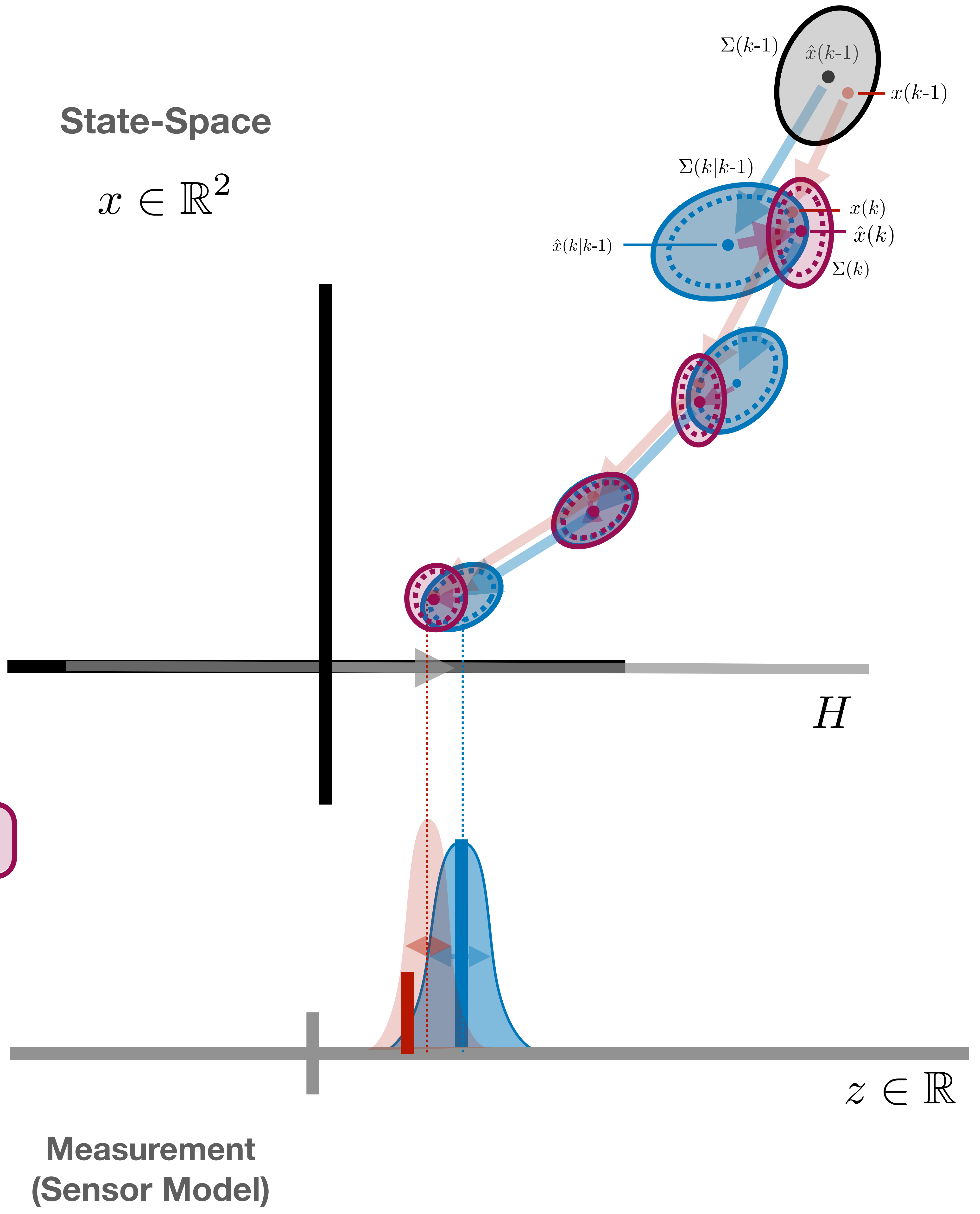


Definite (Symmetric) Matrices - Reference/Review

Quadratic Form: $f(x) = x^T Q x \quad Q \in \mathbb{R}^{n\times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition			Analogy	Eigenvalues
Positive definite:	PD	$Q \succ 0$	$x^TQx > 0$	$\forall\ x$	$x \neq 0$	…positive orthant	$\lambda_i > 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Positive semi-definite	PSD	$Q \succeq 0$	$x^TQx \geq 0$	$\forall\ x$		…positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative-definite	ND	$Q \prec 0$	$x^TQx < 0$	$\forall\ x$	$x \neq 0$	…negative orthant	$\lambda_i < 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Negative semi-definite	NSD	$Q \preceq 0$	$x^TQx \leq 0$	$\forall\ x$		…negative orthant w/ boundary	$\lambda_i \leq 0 \quad \lambda_i \in \mathrm{eig}(Q)$
Indefinite:			$x^TQx > 0$ $x^TQx < 0$	some x some x		…the rest of the space	

Eigenvalue condition proof:

…consider eigenvector coordinates

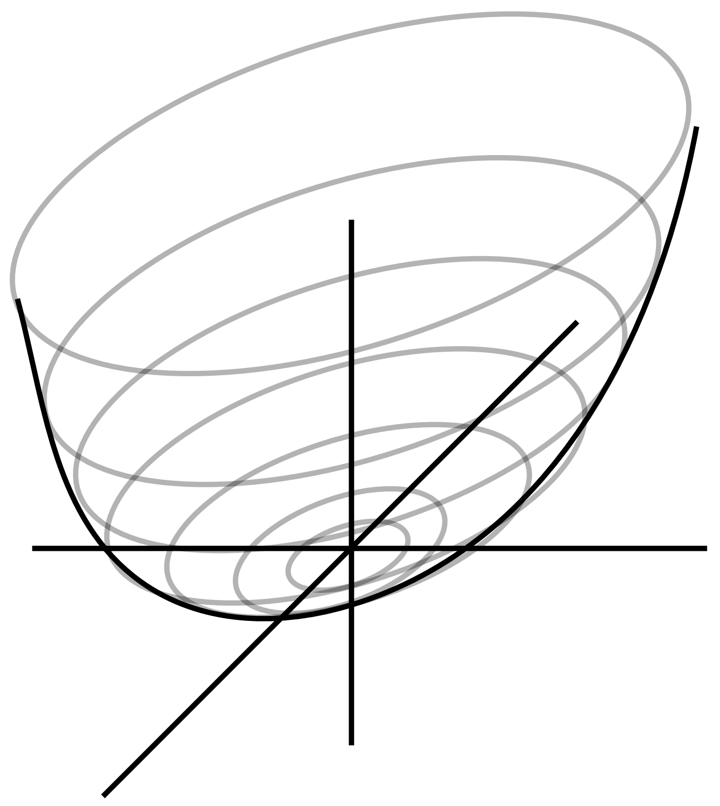
$$x = Vx'$$

since V is invertible… $\forall\ x \iff \forall\ x'$

$$x^TQx = xVDV^Tx = x'^TDx' = \sum_i \lambda_i x_i'^2$$

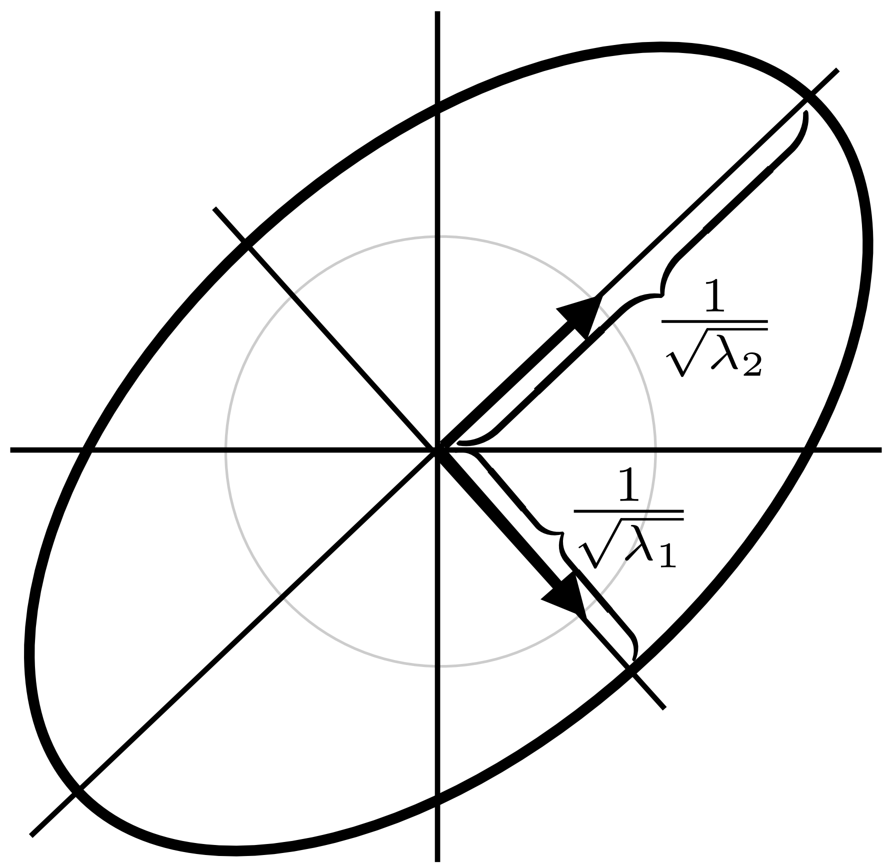
$$\sum_i \lambda_i x_i'^2 > 0 \quad \forall x',\ x \neq 0 \iff \lambda_i > 0 \quad \forall\ \lambda_i \in \mathrm{eig}(Q)$$

Surfaces: $Q \succ 0$



surface

$f(x) = x^TQx = 1$



level sets

$$Q = VDV^T = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \qquad ||v_i||_2 = 1$$

$$f\left(\frac{1}{\sqrt{\lambda_1}} v_1\right) = \frac{1}{\sqrt{\lambda_1}} v_1^T Q v_1 \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \frac{1}{\sqrt{\lambda_1}}$$

$$= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1$$

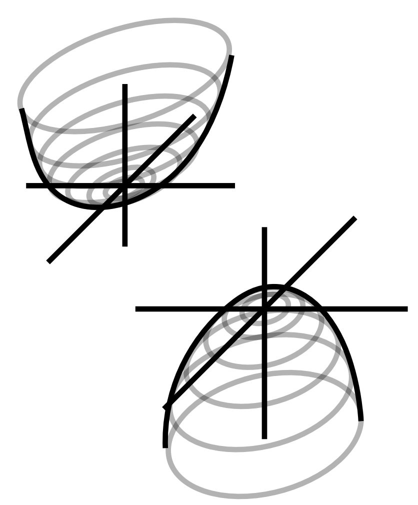
Quadratic Form - Level Sets in 3D - (for fun)

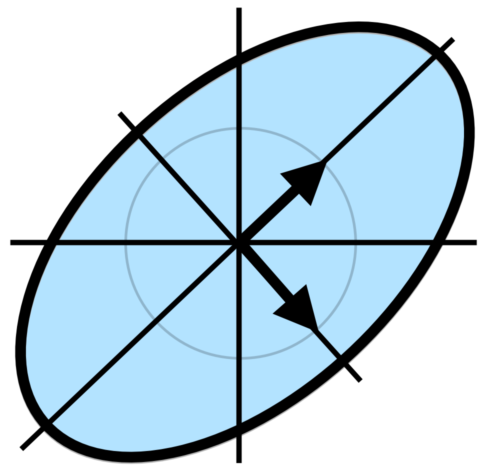
Quadratic Form: $f(x) = x^T Q x \qquad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

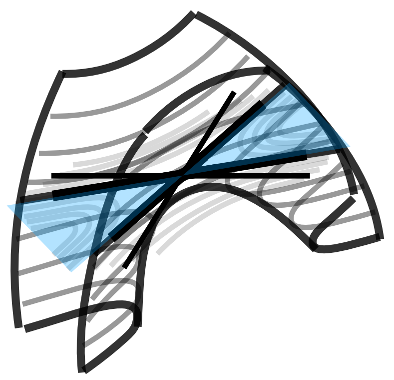
Definite Matrices (Positive or Negative)

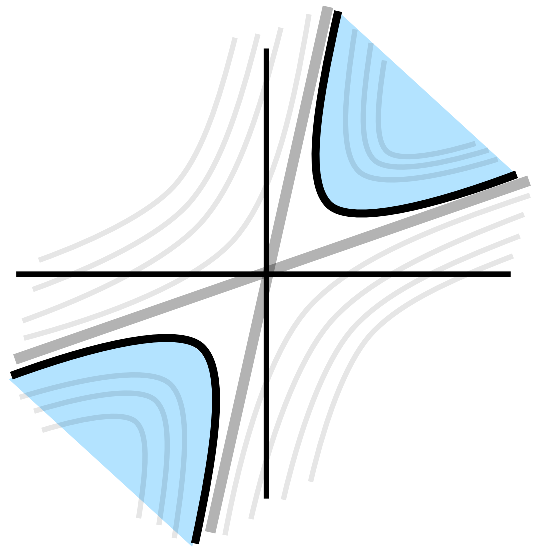
Indefinite

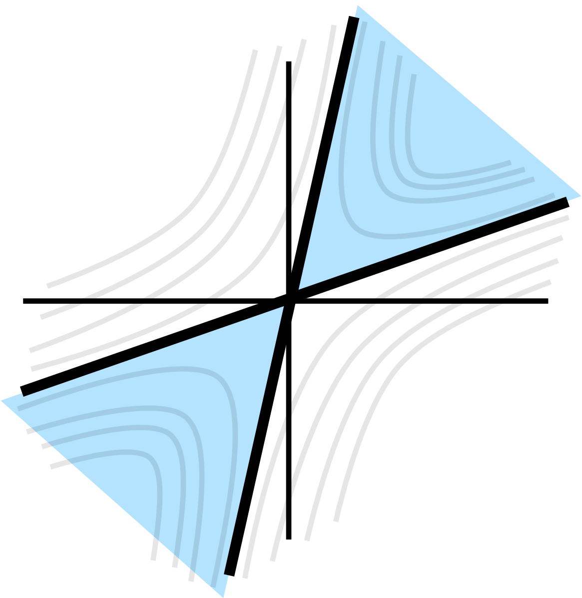
2D

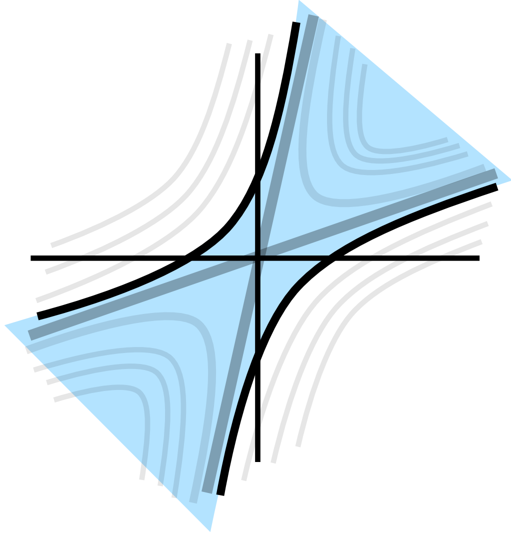




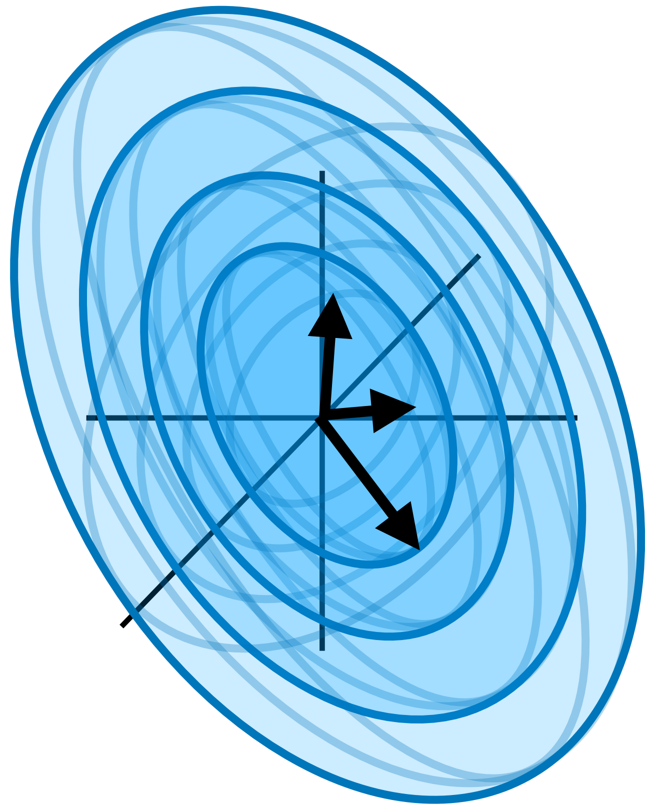


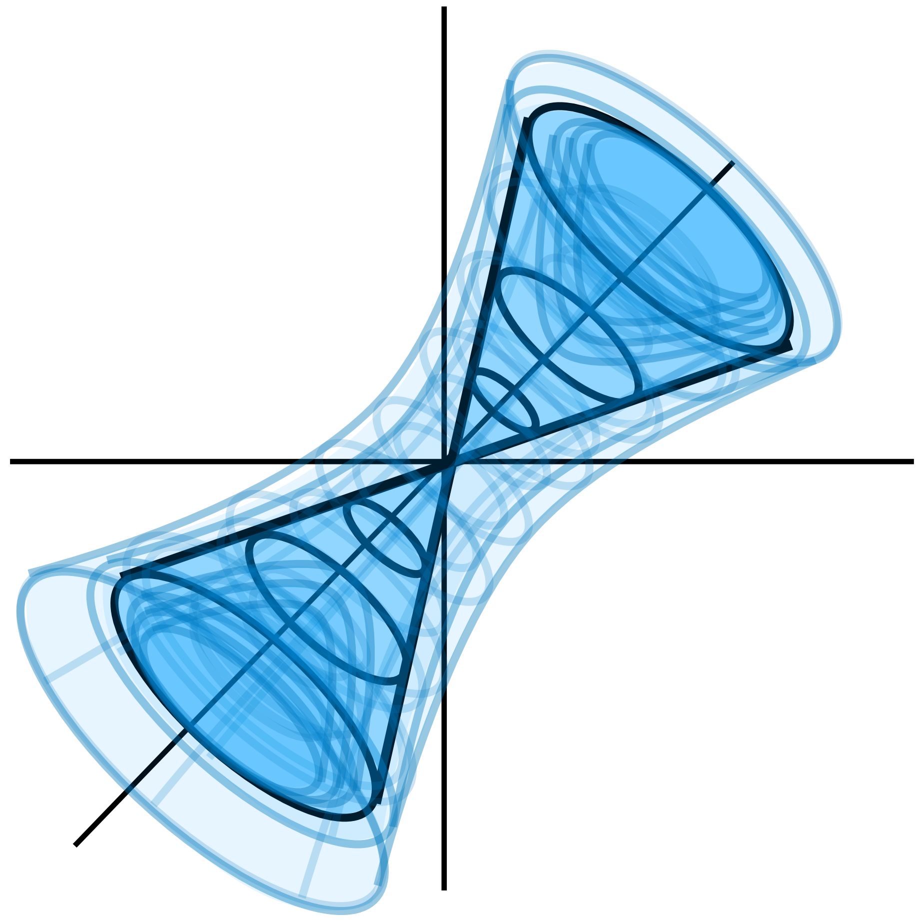


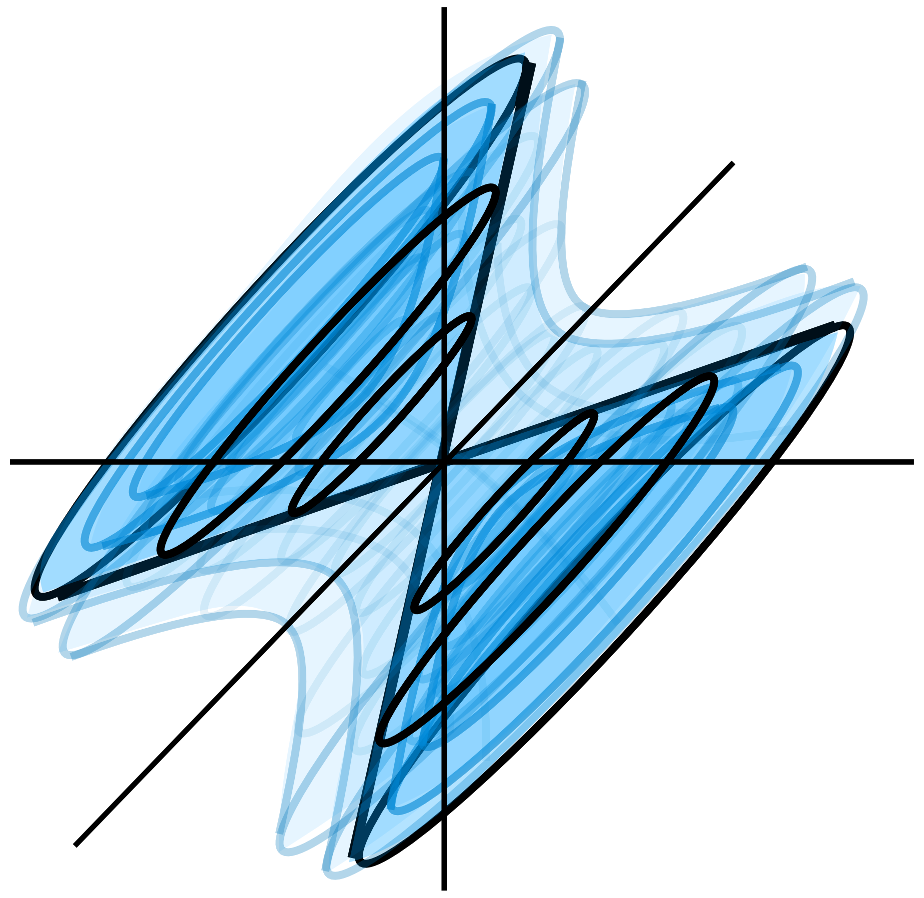




3D







…all positive or all negative eigenvalues

Eigenvalues: two negative, one positive

…expand 1D negative eigenvector into an ellipse…

Eigenvalues: two negative, one positive

…expand 1D positive eigenvector into an ellipse…