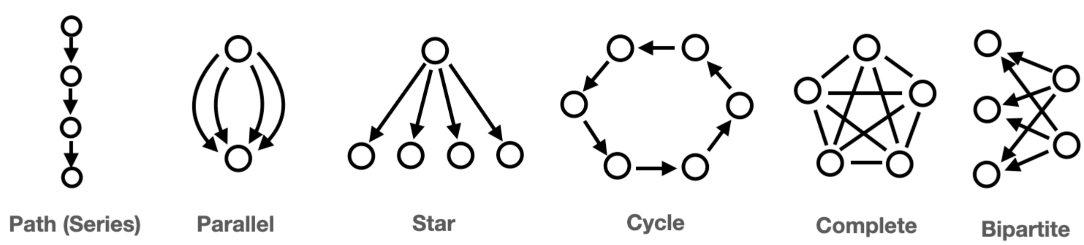
AA597 - Network Dynamics - Spring 2022

Homework 1

1. **Graph Matrices**

   Consider the graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ shown below



   (a) Write the following matrices for each graph in the image above.

   - Incidence matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times \mathcal{E}|}$
   - (Undirected) Adjacency matrix: $A \in \mathbb{R}^{|\mathcal{V}| \times \mathcal{V}|}$
   - (Undirected) Laplacian: $L \in \mathbb{R}^{|\mathcal{V}| \times \mathcal{V}|}$
   - Directed Adjacency matrix: $A_{\text{in}} \in \mathbb{R}^{|\mathcal{V}| \times \mathcal{V}|}$
   - In-degree Laplacian: $L_{\text{in}} \in \mathbb{R}^{|\mathcal{V}| \times \mathcal{V}|}$

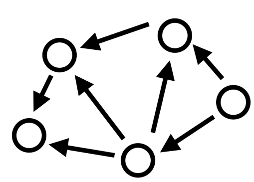
   For each graph, choose node and edge orderings that make the structure easier to see. (If an edge orientation is not given, pick one.) Pay attention to how the structure of each graph is expressed in each matrix. Be careful how the definition of these matrices are different for directed vs undirected graph.

   (b) For the Laplacian and adjacency matrices from above, compute their eigenvalues and eigenvectors. What structure do you notice?

   Note: it's not always easy to notice relevant structure from numerical computations of things like eigenvectors so don't stress about this part.

2. **Incidence Matrix Decomposition**

   Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ shown below



(a) Write the incidence matrix $D \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ for this graph.
(You can select appropriate orderings for the vertices and edges.)

(b) Write a decomposition of D as

$$D = T\begin{bmatrix} I & M \end{bmatrix}$$

where T is the incidence matrix for a spanning forest. (Note you will need to order the nodes and edges correctly for this form to work exactly as shown.)

(c) Write bases for $\mathcal{R}(D)$, $\mathcal{R}(D^T)$, $\mathcal{N}(D)$, and $\mathcal{N}(D^T)$,
ie. write bases for the range and nullspaces of D and D^T.

(d) Prove the above are bases for the given spaces. You can make these arguments numerically if you want.

(e) Generalize these basis constructions to arbitrary graphs and prove each basis is linearly independent and spans the desired space. Note: there are eight proofs to do: linear independence and span for each subspace.

Hints:. Probably the most compact characterization of linear independence (of the columns of A) is the following

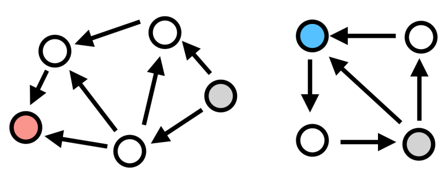
$$Ax = 0, \quad \Rightarrow \quad x = 0$$

(Think about why this is the case.). Many linear independence proofs become easier using this definition. Feel free to ask for more hints in class.

3. **Flow and Cut Spaces**

(a) **Edge Flow Space (nullspace of D)**

Write a source-sink vector $S \in \mathbb{R}^{|\mathcal{V}|}$ that indicates 2 units of mass-flow entering the network at each of the grey nodes and 2 units of mass exiting the network at each of the red and blue nodes.



Note that we can define the space of possible edge flows consistent with this source-sink vector as

$$\mathcal{X} = \left\{ x \;\middle|\; Dx = S,\ x \in \mathbb{R}^{|\mathcal{E}|} \right\}$$

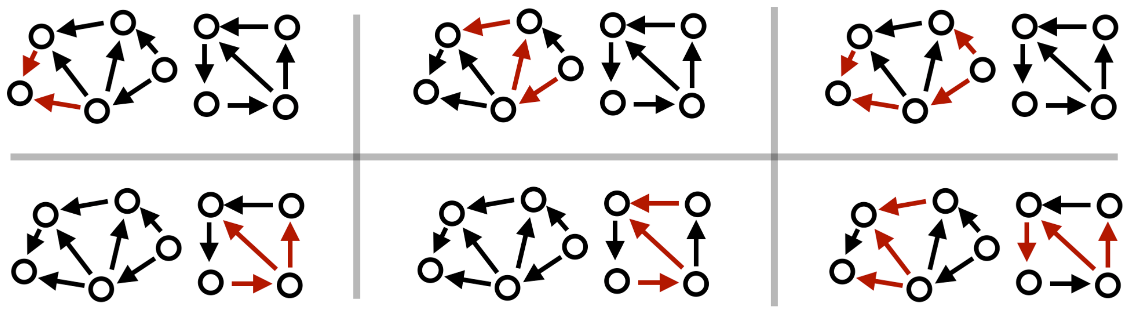
BY HAND: Rewrite the set $\mathcal{X}$ in the form

$$\mathcal{X} = \left\{ x \;\middle|\; x = \bar{x} + Cz,\ x \in \mathbb{R}^{|\mathcal{E}|},\ z \in \mathbb{R}^{|\mathcal{C}|} \right\}$$

ie. find a specific solution $\bar{x} \in \mathbb{R}^{|\mathcal{E}|}$ and a basis for the nullspace of D, $C \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{C}|}$ where $|\mathcal{C}|$ is the dimension of the cycle space.

(b) **Edge Cut Space (range of D^T)**

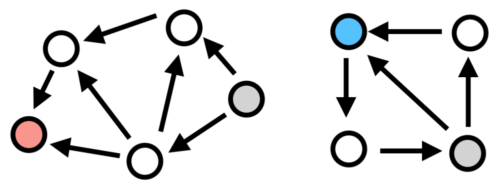
For each of the graphs given in the figure



find vectors $w \in \mathbb{R}^{|\mathcal{V}|}$ such that $w^T D$ gives signed indicator vectors for the edges highlighted in red. Note that the elements of each w should be 1's or 0's (or -1's are fine too).

4. **Counting Walks**

Consider the graph from above



(a) Use powers of the adjacency matrix to count the number of 3 step, 4 step, and 10 step walks from the grey nodes to the red and blue nodes respectively (ignoring edge direction).

(b) Use sums of powers of the adjacency matrix to count the number of walks from the grey nodes to the red and blue nodes respectively with less than or equal to 10 steps (ignoring edge direction).

5. **Laplacian & Adjacency**

(a) Show that for a d-regular graph the (undirected) Laplacian and adjacency matrices have the same eigenvectors and each eigenvalue of the Laplacian has the form $d - \lambda$ where λ is an eigenvalues of the adjacency matrix.

6. **Node and Edge Laplacians**

(a) (Undirected) Laplacians can sometimes be thought of as "relative shape" matrices for the columns and rows of the incidence matrix. Show that the node Laplacian $L = DD^T$ stays the same if you rotate the rows and that the edge Laplacian $L_e = D^T D$ stays the same if you rotate the columns.

(b) Relabeling the nodes or edges of a graph corresponds to multiplying the incidence matrix on the left or right by a permutation matrix. Show that relabeling the edges does not change the node Laplacian $L = DD^T$ and that relabeling nodes does not change the edge Laplacian $L_e = D^T D$.

(c) Show that the rank of $L = DD^T$ is the number of nodes minus the number of connected components.

(d) For a graph with 10 nodes, 100 edges, and three connected components, what is the dimension of the nullspace of $L = DD^T$? What is the dimension of the nullspace of $L_e = D^T D$?

(e) Eigenvectors of the node Laplacian $L = DD^T$ can be thought of as vibration modes of the nodes of the graph. If v is an eigenvector of $L = DD^T$ with eigenvalue λ, show that $D^T v$ is an eigenvector of $L_e = D^T D$ also with eigenvalue λ. (Hint: this construction is the heart of the derivation of the SVD). If v can be interpreted as a vibration mode of a graph, what is an interpretation for $D^T v$ (no wrong answers).

7. **Laplacian Eigenvalues**

We discussed how the (undirected) Laplacian is positive semi-definite with at least one eigenvalue of 0 corresponding to an eigenvector of all 1's, $\mathbf{1}$. Denote the ordered spectrum of the Laplacian as

$$0 \leq \lambda_2(L) \leq \cdots \leq \lambda_{\max}(L)$$

(a) Show that $\lambda_2(L)$ is strictly greater than 0 if and only if the graph has one connected component.

There are useful characterizations of other eigenvalues based on optimization ideas. For example $\lambda_{\max}(L)$ can be written as

$$\lambda_{\max}(L) = \max_{||x||_2=1} x^T L x$$

(Think about why this might be the case.)

We can also characterize $\lambda_2(L)$ as

$$\lambda_2(L) = \min_{\substack{||x||_2=1 \\ \mathbf{1}^T x=0}} x^T L x$$

(b) Show that the maximum eigenvalue of the Laplacian, $\lambda_{\max}(L)$, is greater than the maximum degree of the graph.

(c) Show that

$$\lambda_2(L) \leq \min_{i,j \in \mathcal{V}, j \notin \mathcal{N}_i} \frac{1}{2}(d_i + d_j)$$

(d) Show that for any graph with two nodes of minimum degree that are not adjacent, $\lambda_2(L)$ is less than the minimum degree of the graph. (This is true in general, but harder to prove.)