Graph Structures & Matrices

Algebraic Graph Theory

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Shahriar Talebi

Spring 2022 - Dan Calderone

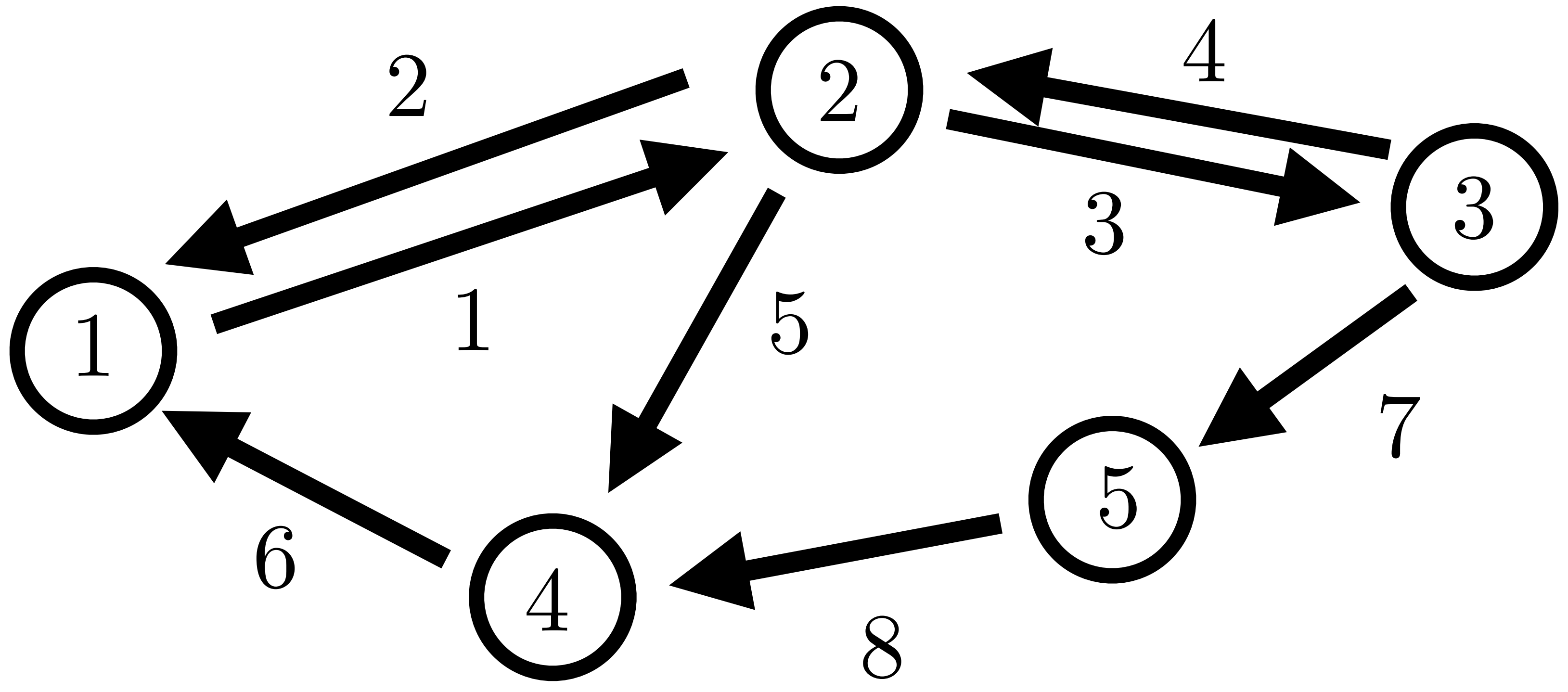
Graphs

Graph:

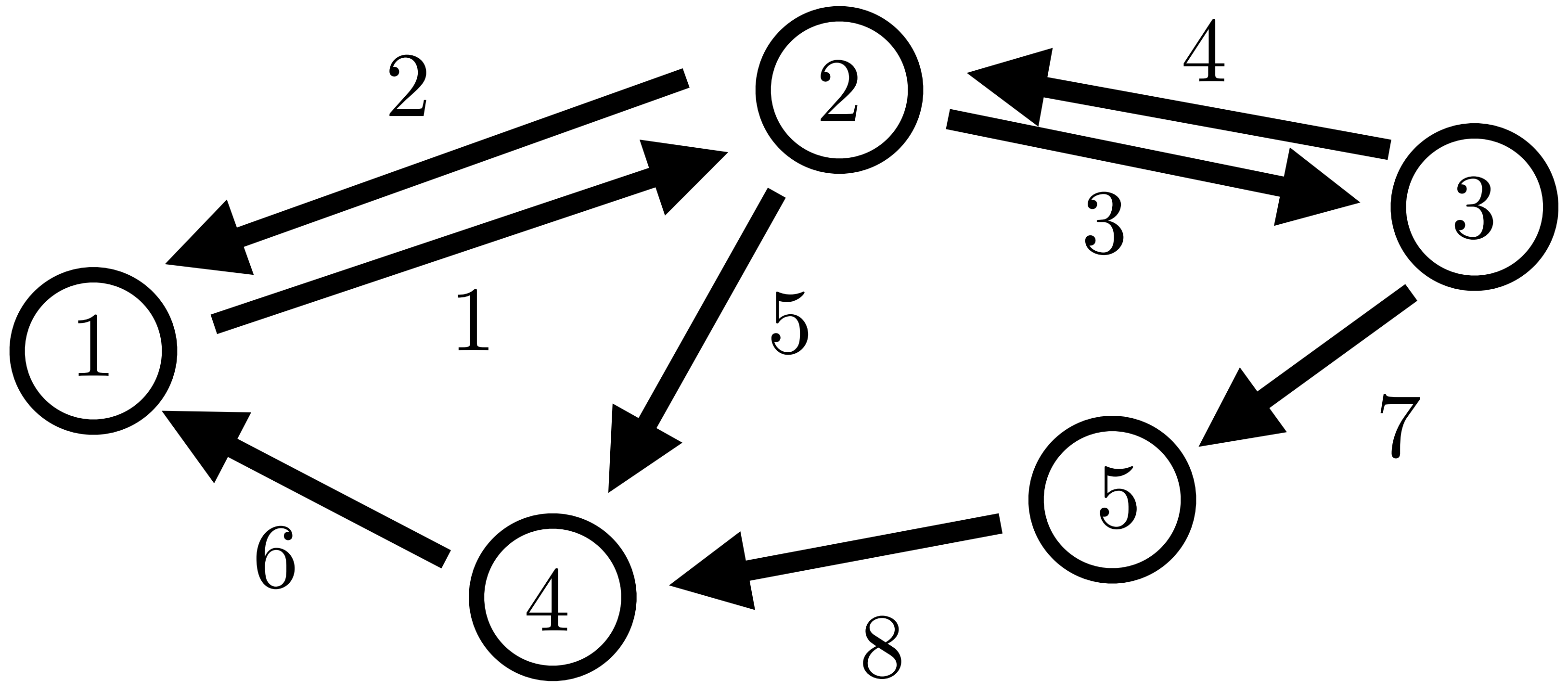
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$



Graphs



Graph:

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edge e is “incident” to v and v’

Undirected Graphs

$$e = (v, v')$$

Neighborhoods: set of “adjacent” nodes

$$\mathcal{N}_v = \{v' \in \mathcal{V} \mid e = (v, v') \in \mathcal{E}\}$$

degree of vertex $d_v = |\mathcal{N}_v|$

Directed Graphs

$e = (v, v')$ edge e from v to v’

Neighborhoods: set of “adjacent” nodes

$$\mathcal{N}_v^{\text{out}} = \{v' \in \mathcal{V} \mid e = (v, v') \in \mathcal{E}\}$$

$$\mathcal{N}_v^{\text{in}} = \{v' \in \mathcal{V} \mid e = (v', v) \in \mathcal{E}\}$$

$$\mathcal{N}_v = \mathcal{N}_v^{\text{in}} \cup \mathcal{N}_v^{\text{out}}$$

out-degree $d_v^{\text{in}} = |\mathcal{N}_v^{\text{in}}|$

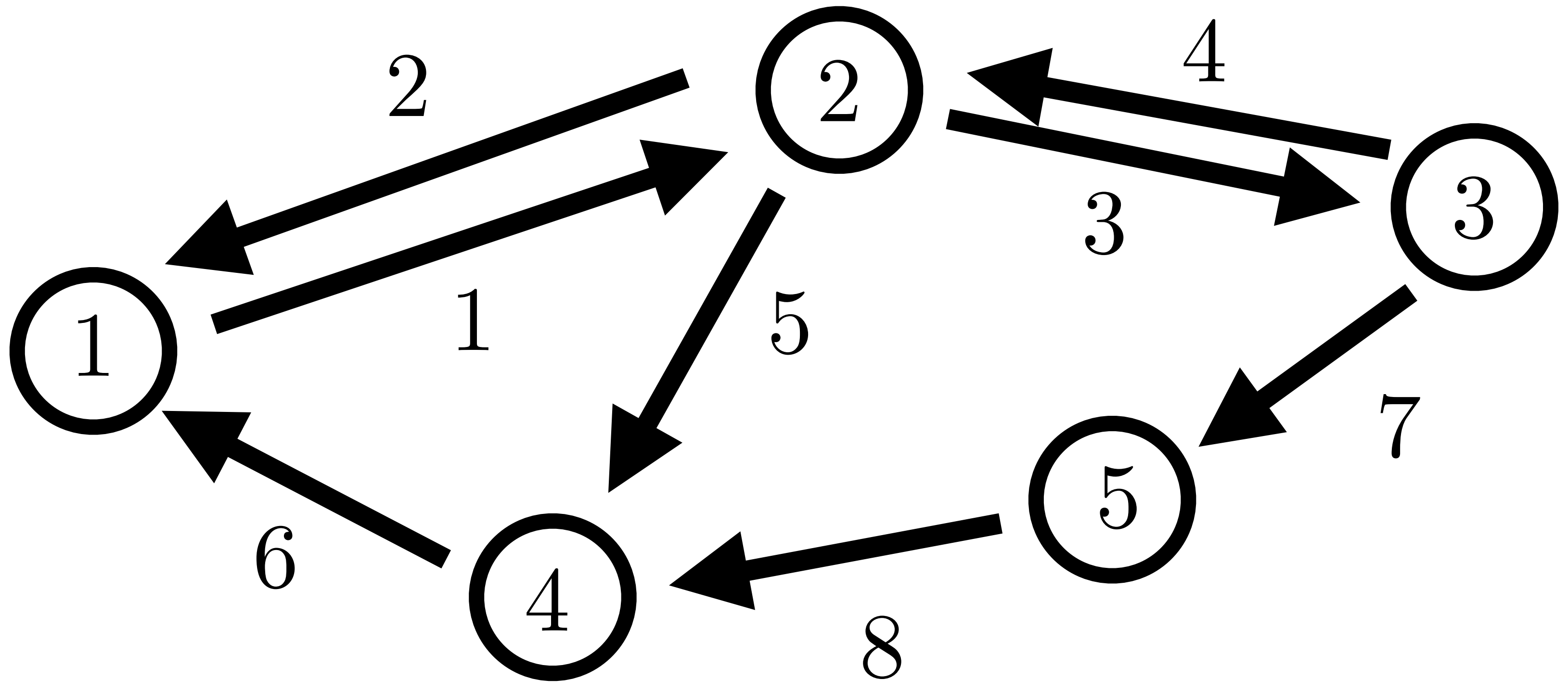
in-degree $d_v^{\text{out}} = |\mathcal{N}_v^{\text{out}}|$

degree $d_v = d_v^{\text{in}} + d_v^{\text{out}}$

Automorphism of Graph

“Relabeling of nodes and edges
that maintains graph structure”

Incidence Matrix



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$$\Big[\ D\ \Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

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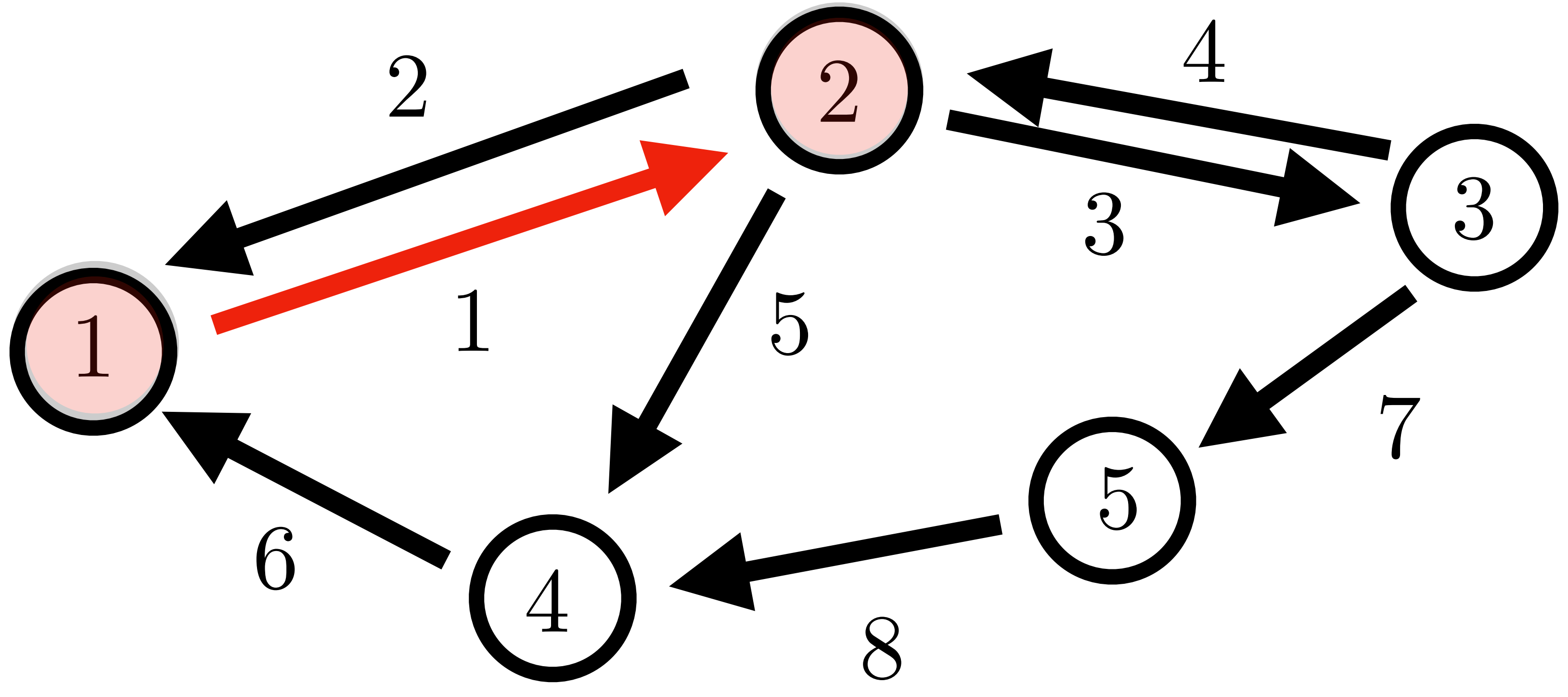
$$\begin{array}{ccc} & \longleftarrow\ \text{edges}\ \longrightarrow & \\ D = & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{matrix} \uparrow \\ \text{vertices} \\ \downarrow \end{matrix} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



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← edges →

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↑ vertices ↓

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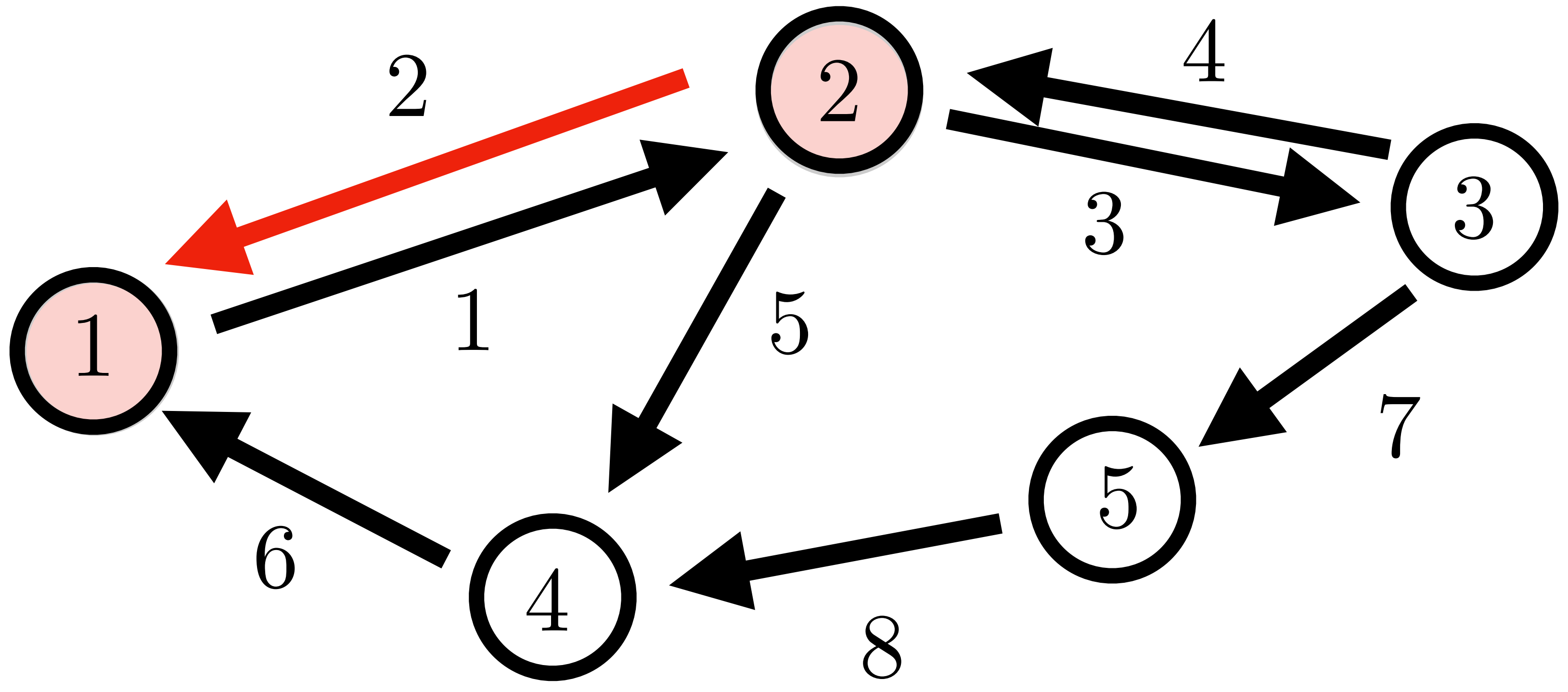

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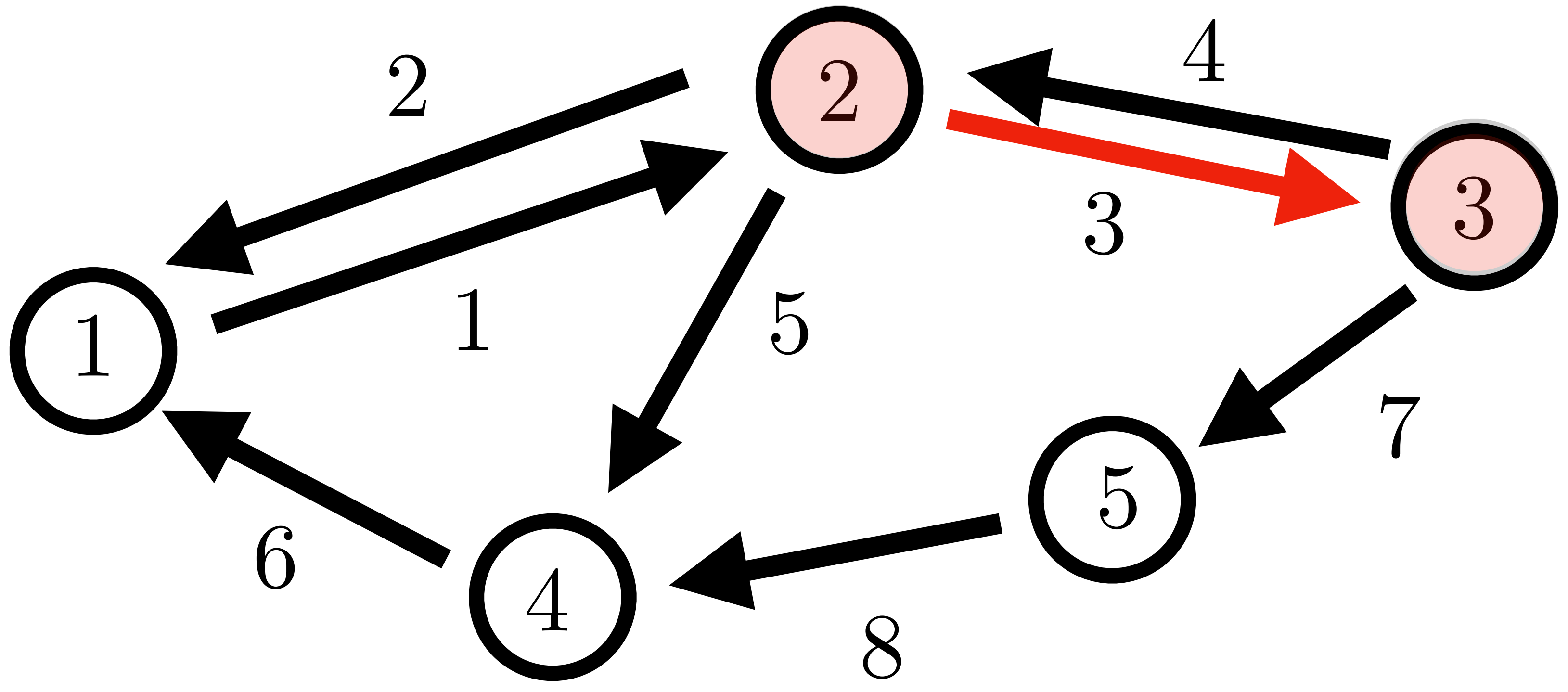

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edges

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vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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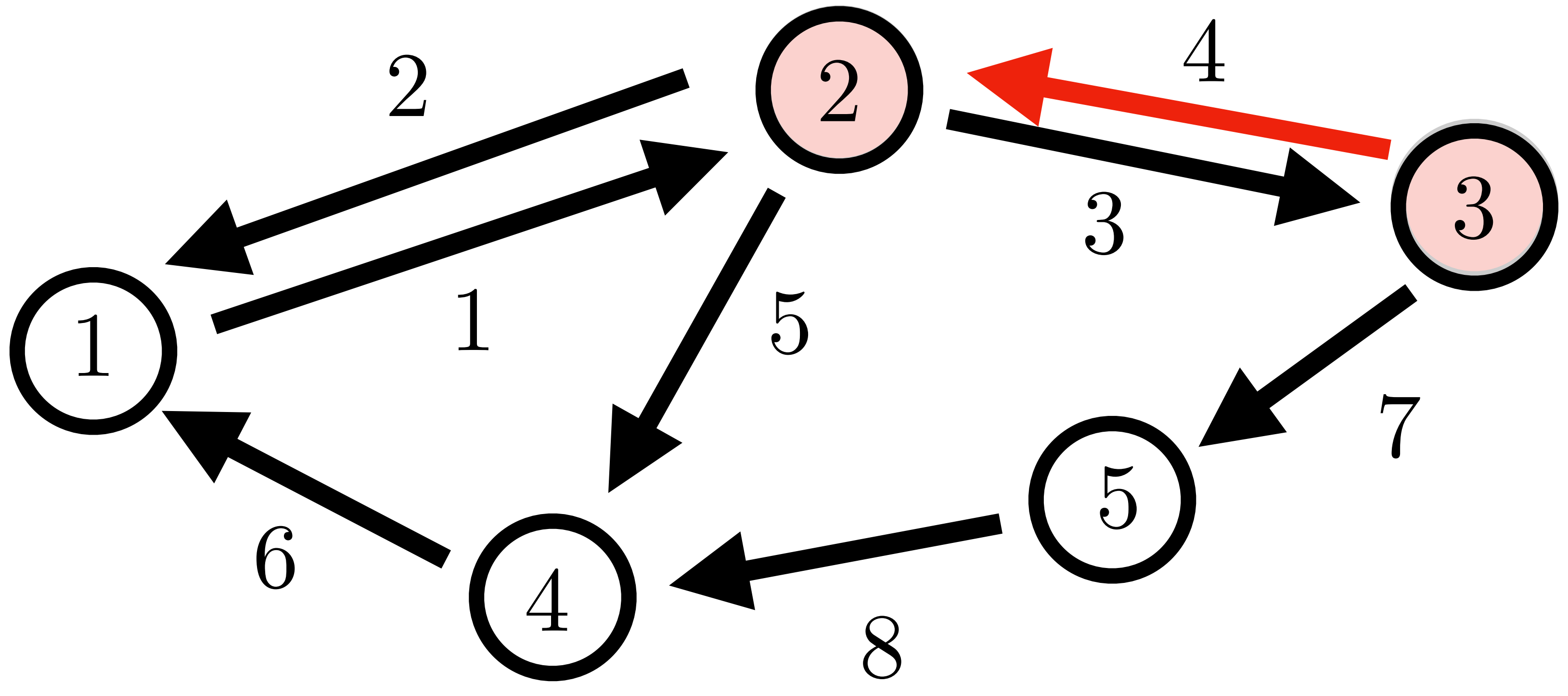

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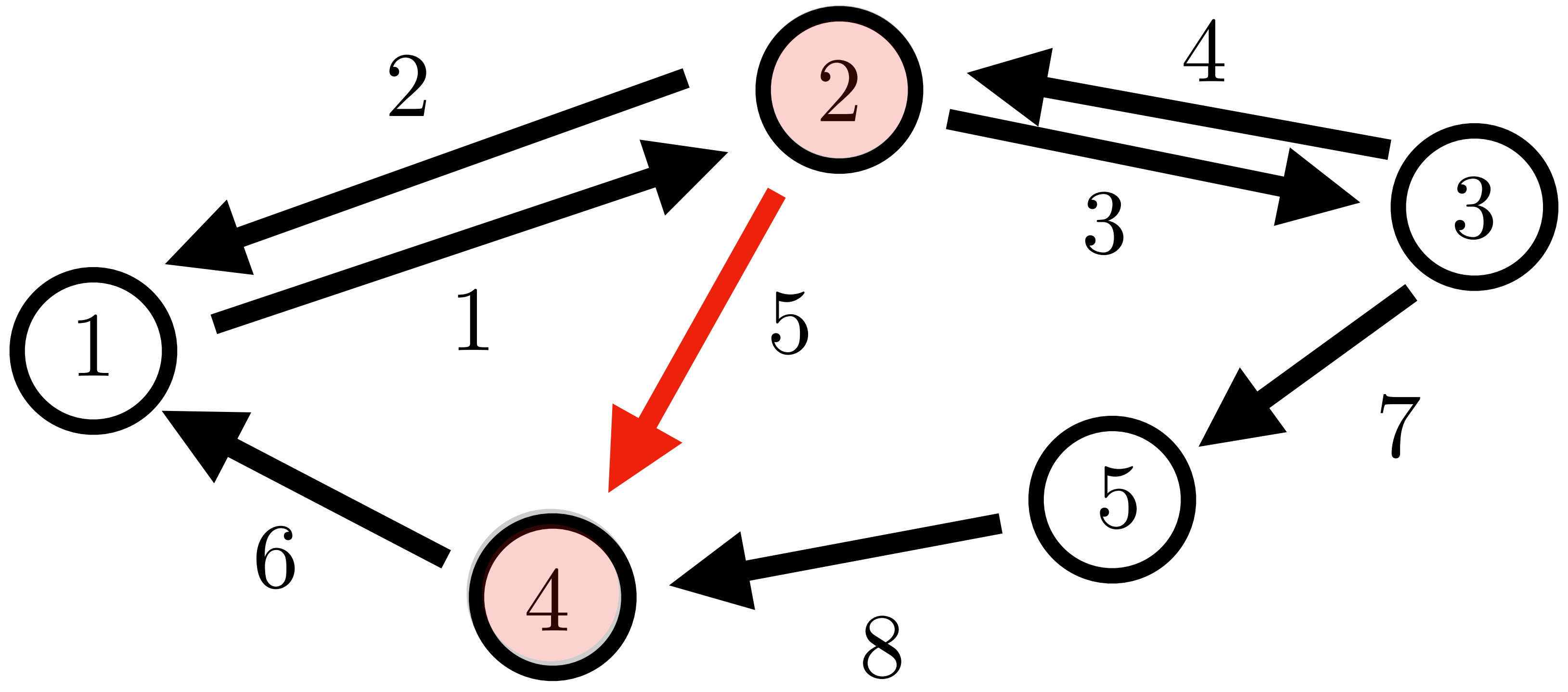

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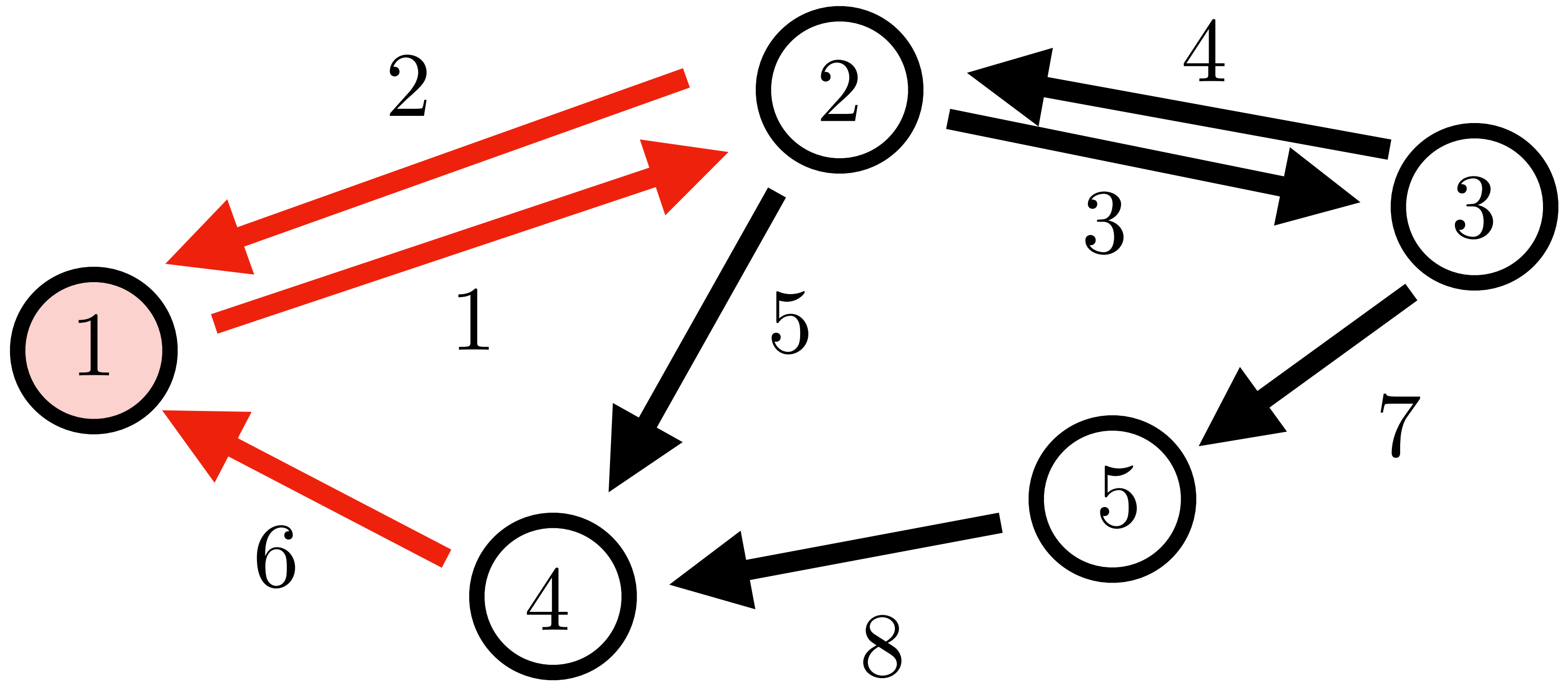

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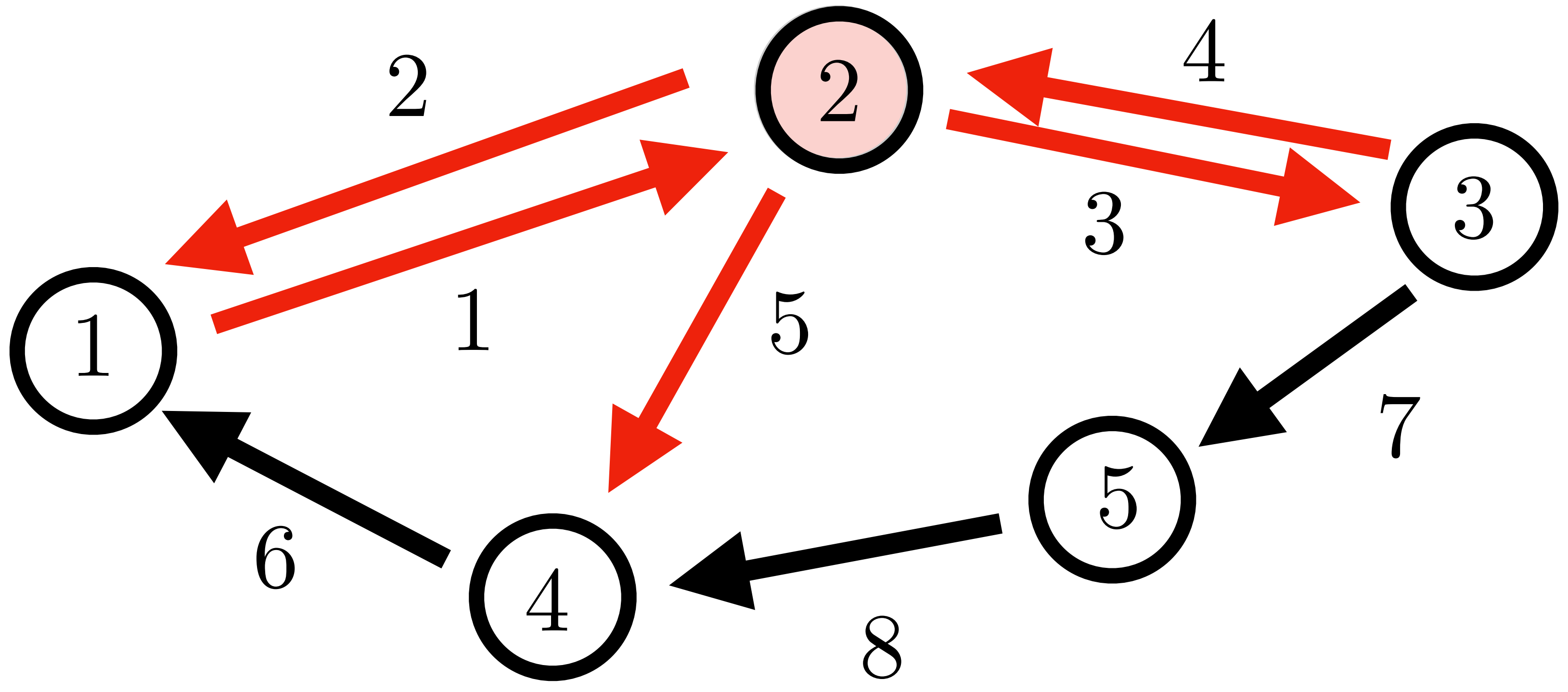

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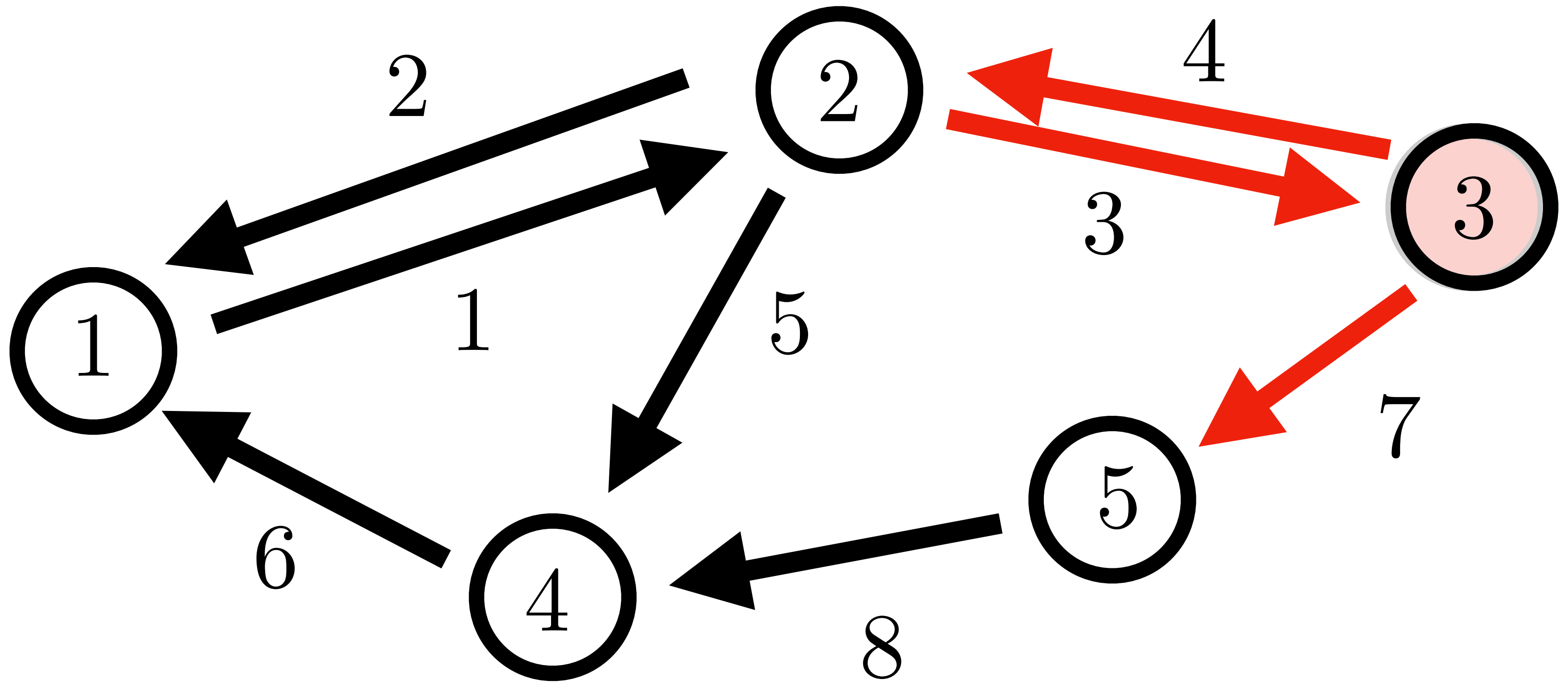

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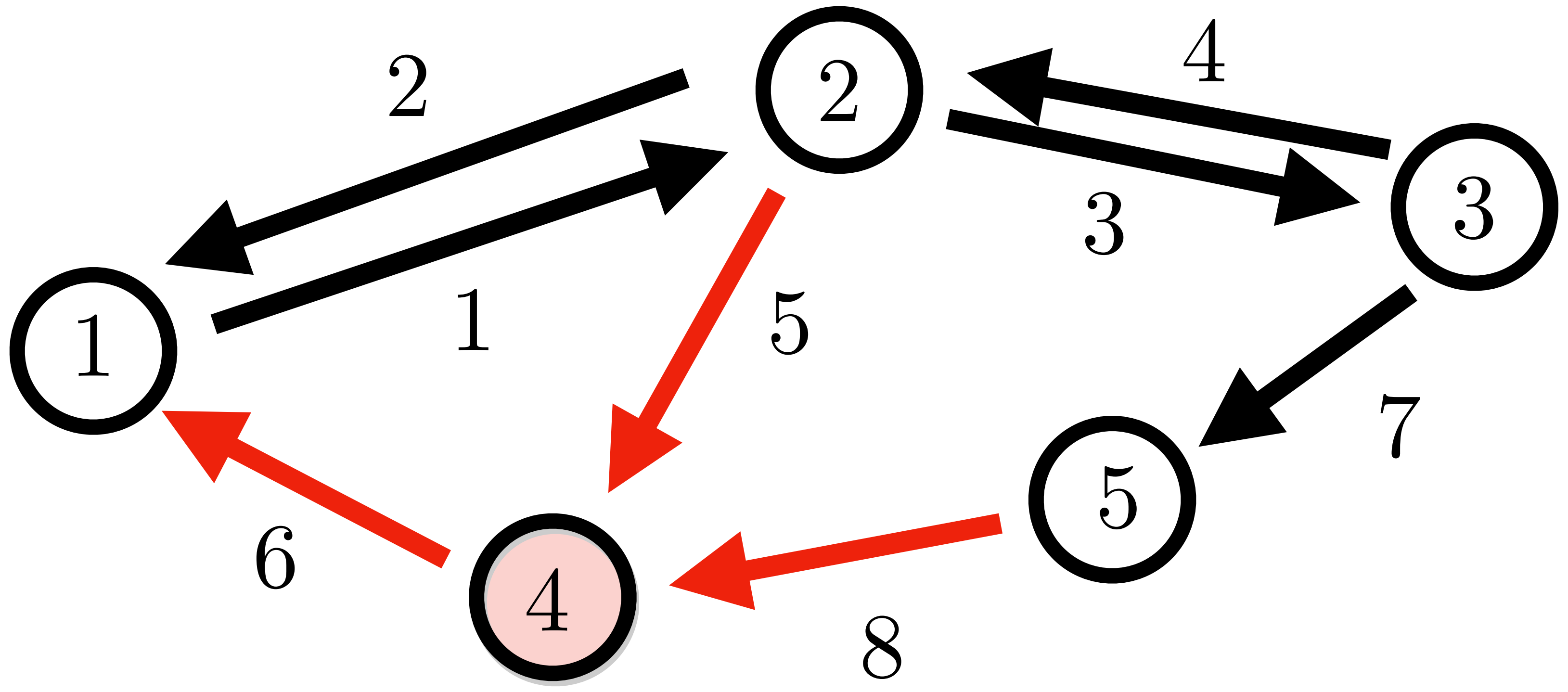

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Incidence Matrix

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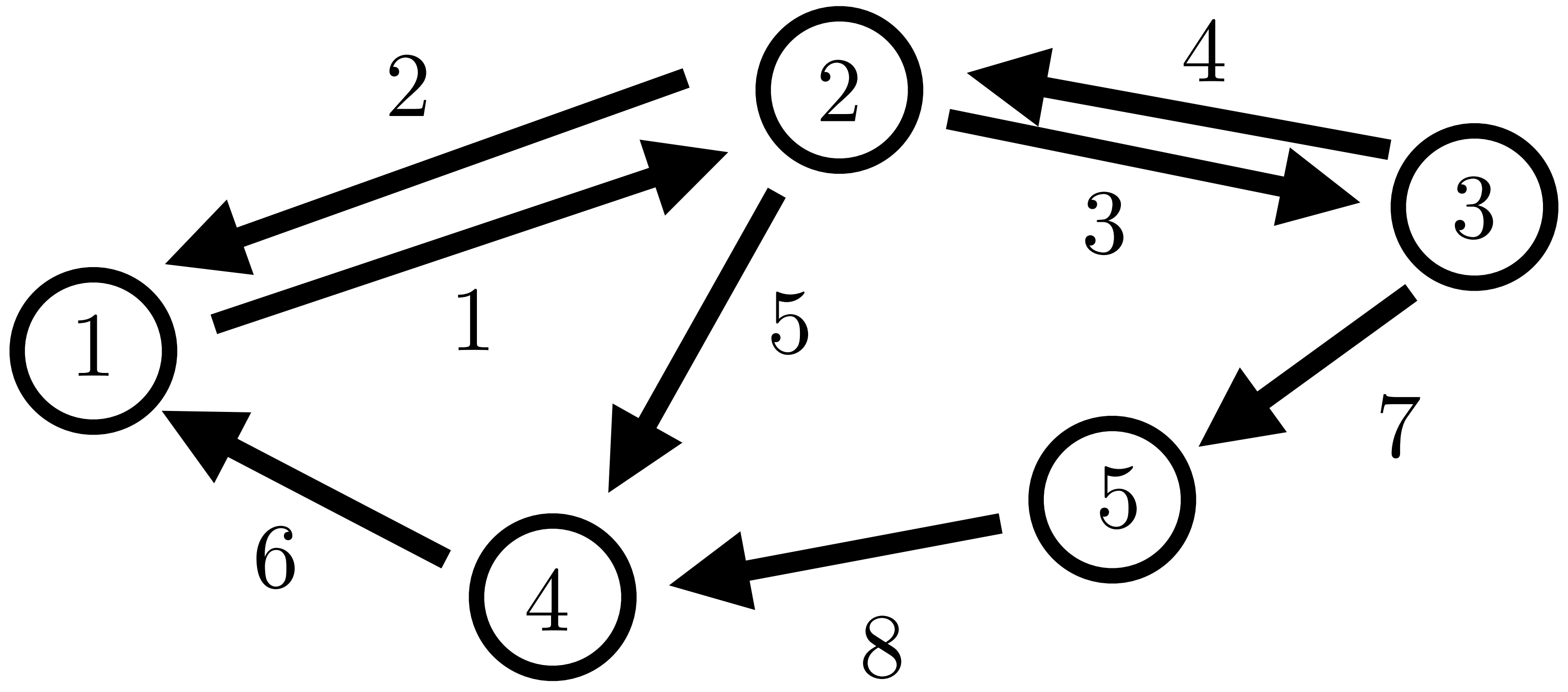
$$\xleftrightarrow{\quad\text{edges}\quad}$$

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \updownarrow \text{vertices}$$



...relabeling nodes **rearrange rows**

...relabeling edges **rearrange columns**

Algebraically: multiply by permutation matrices

P, P' permutation matrices

New Incidence Matrix $D' = PDP'$

Incidence Matrix - Domain

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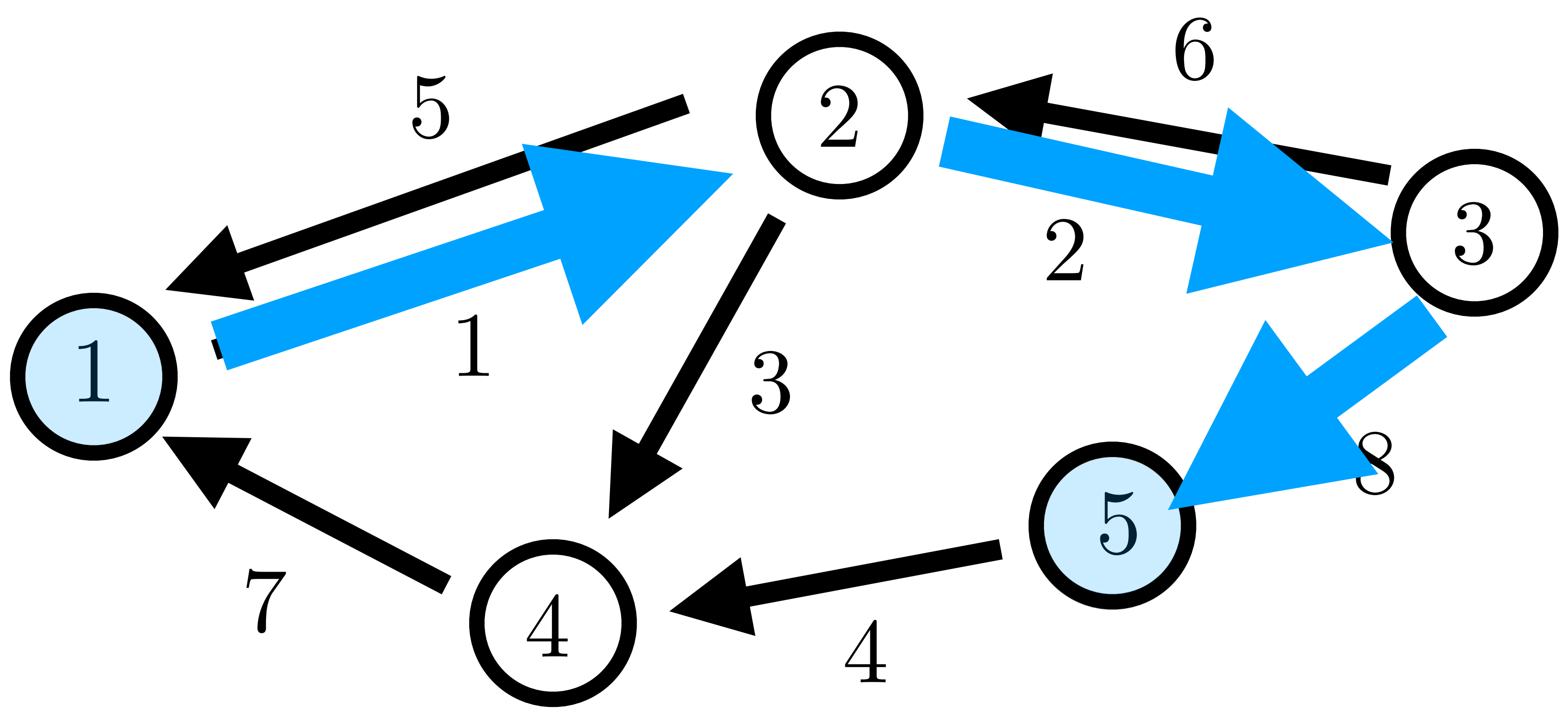

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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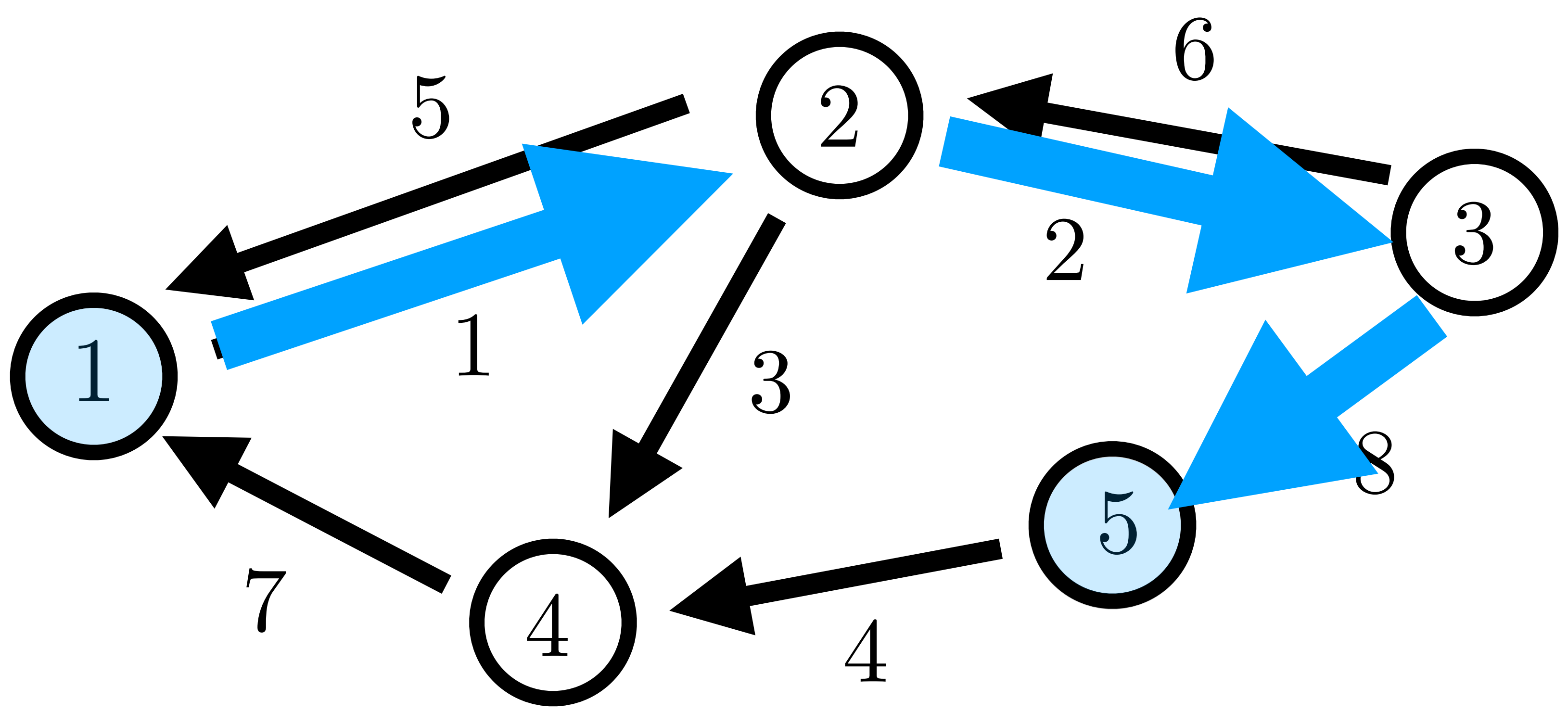
Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Incidence Matrix - Domain



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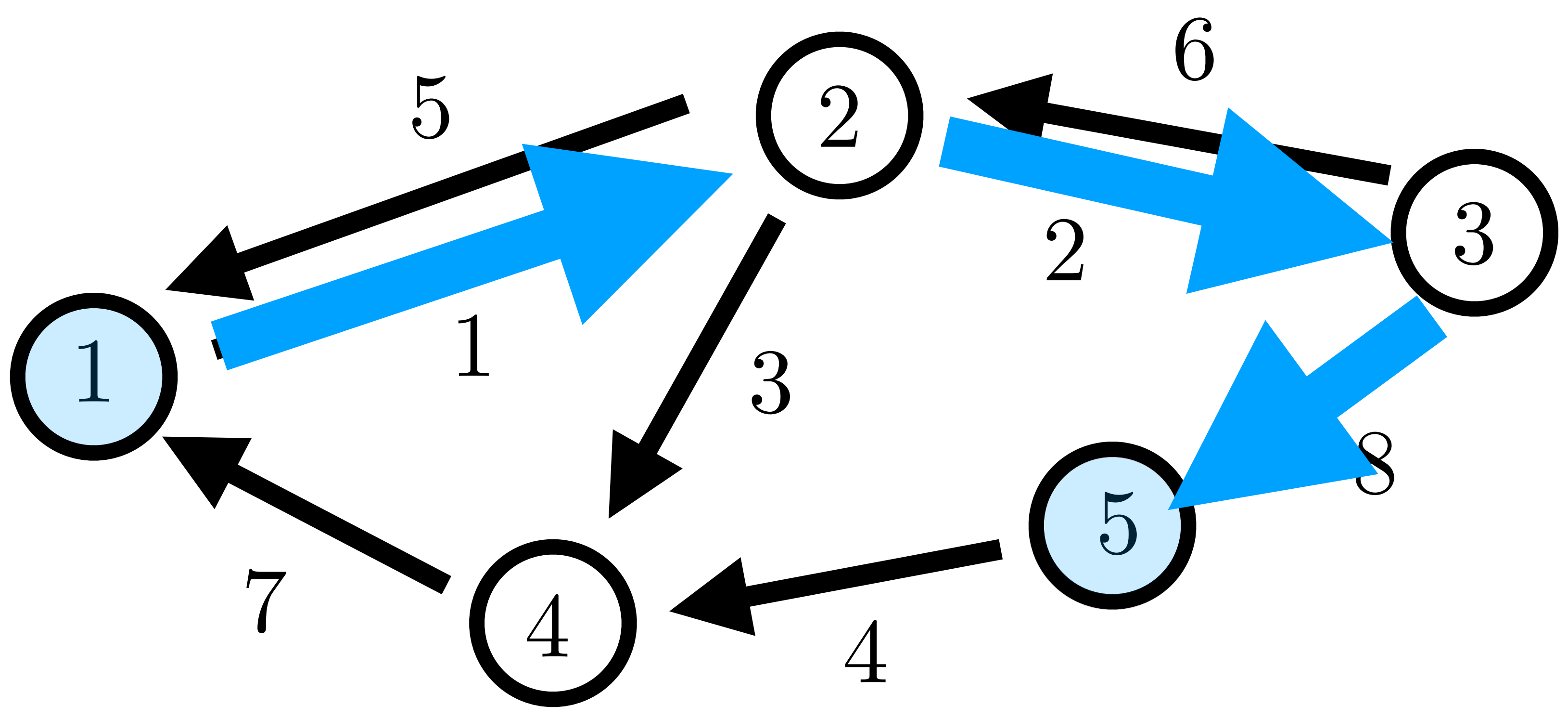
Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ …mass flow on edges

Examples

…fluid flow
…traffic flow
…data flow
…current

Incidence Matrix - Domain



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Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ …mass flow on edges

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ …source-sink on nodes

Examples

…fluid flow

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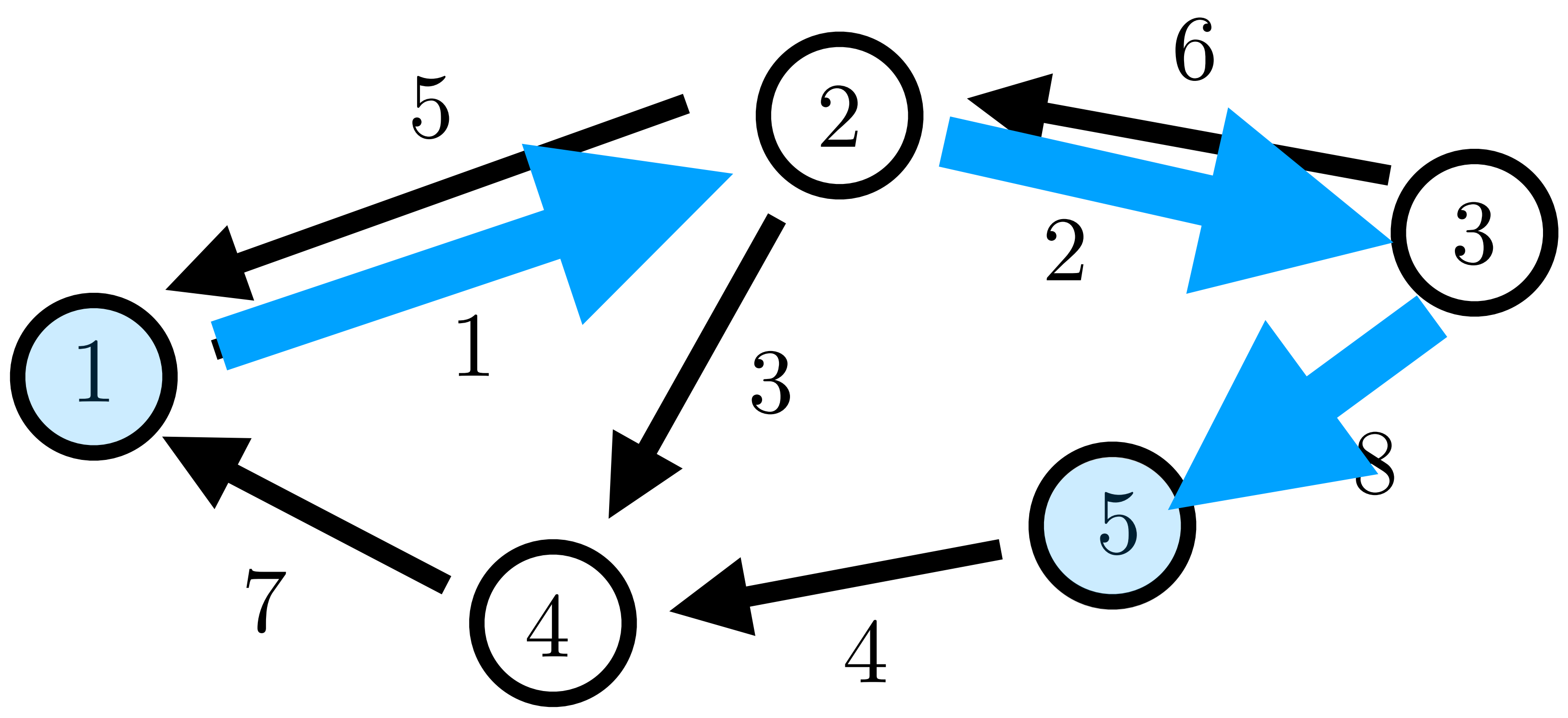
Incidence Matrix - Domain



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Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

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Non-conserved flow

$$S = Dx$$

Edge flow vector

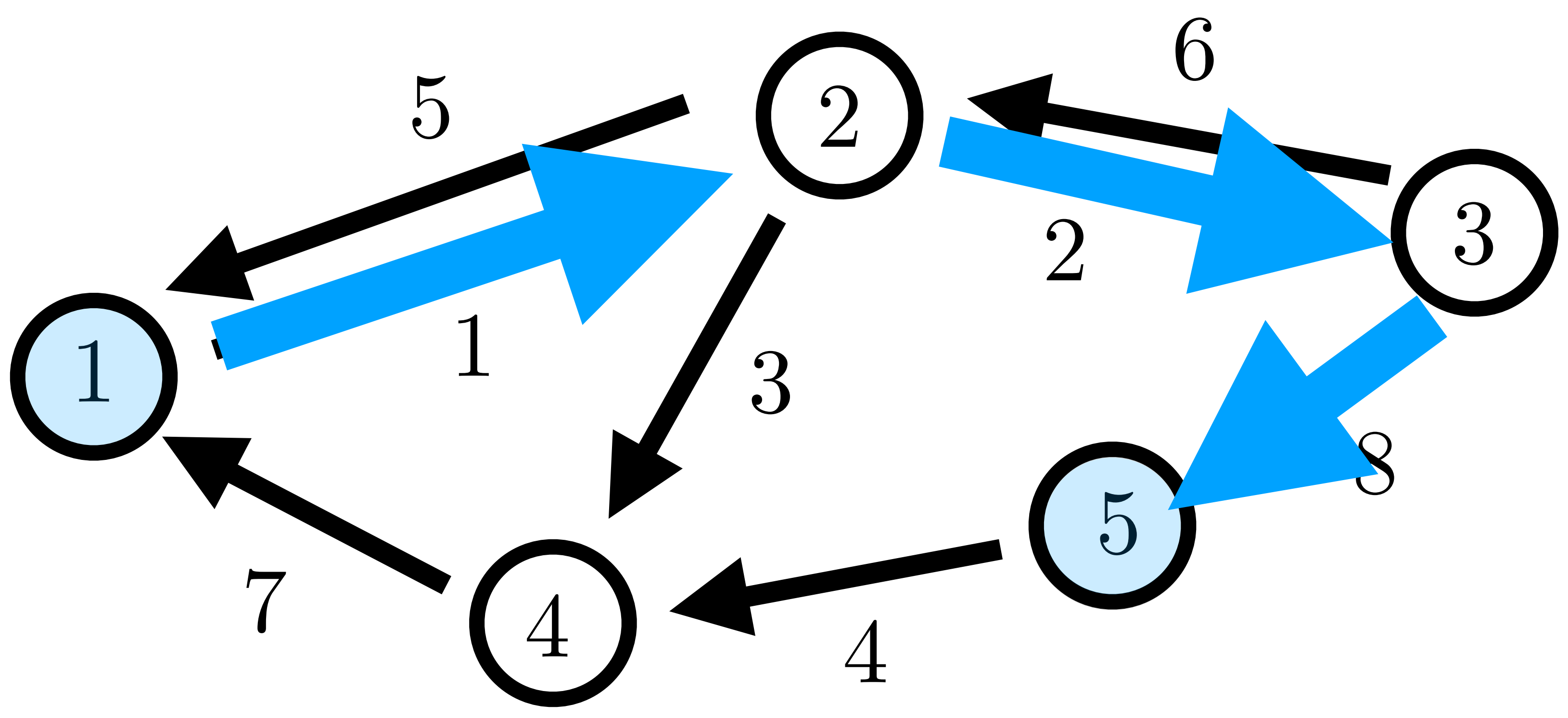
$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Incidence Matrix - Domain



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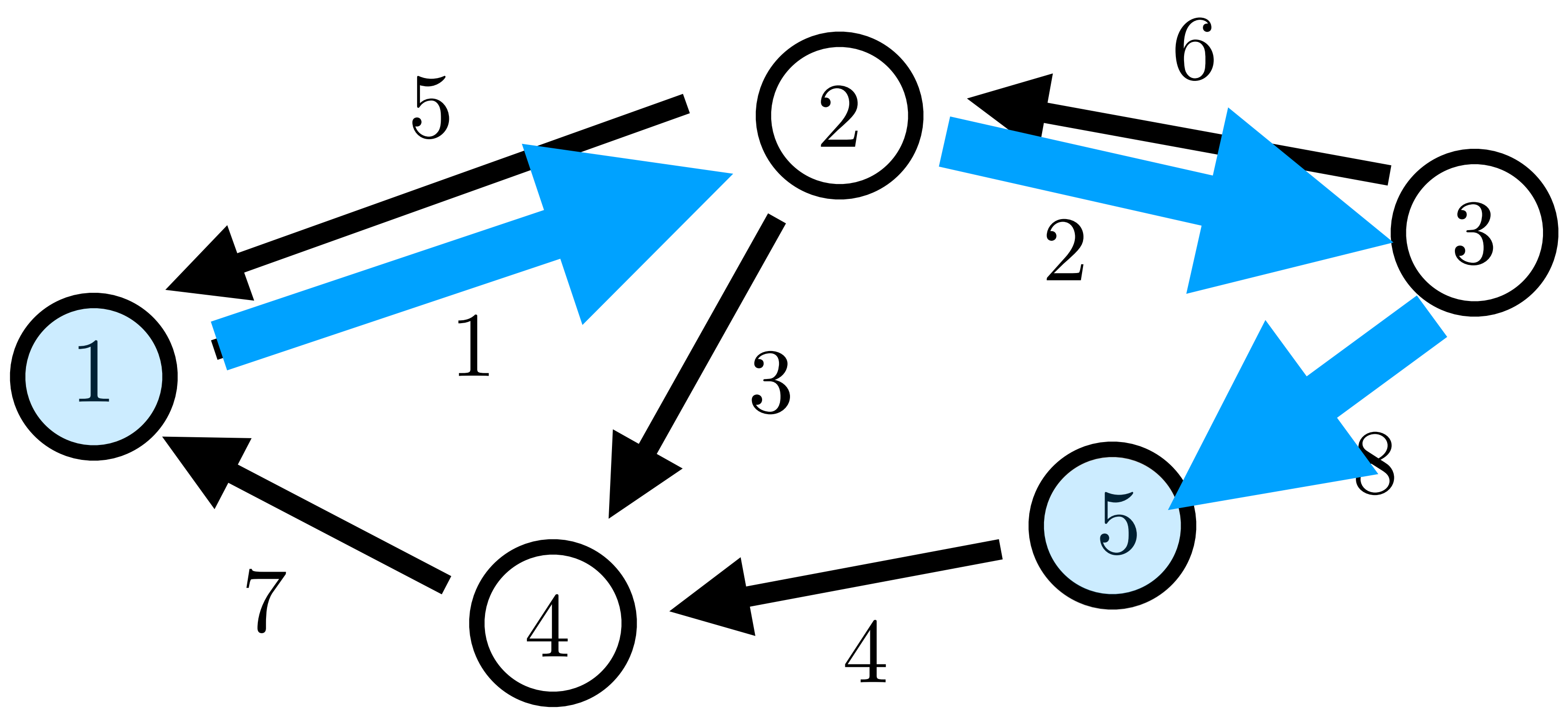
Incidence Matrix - Domain



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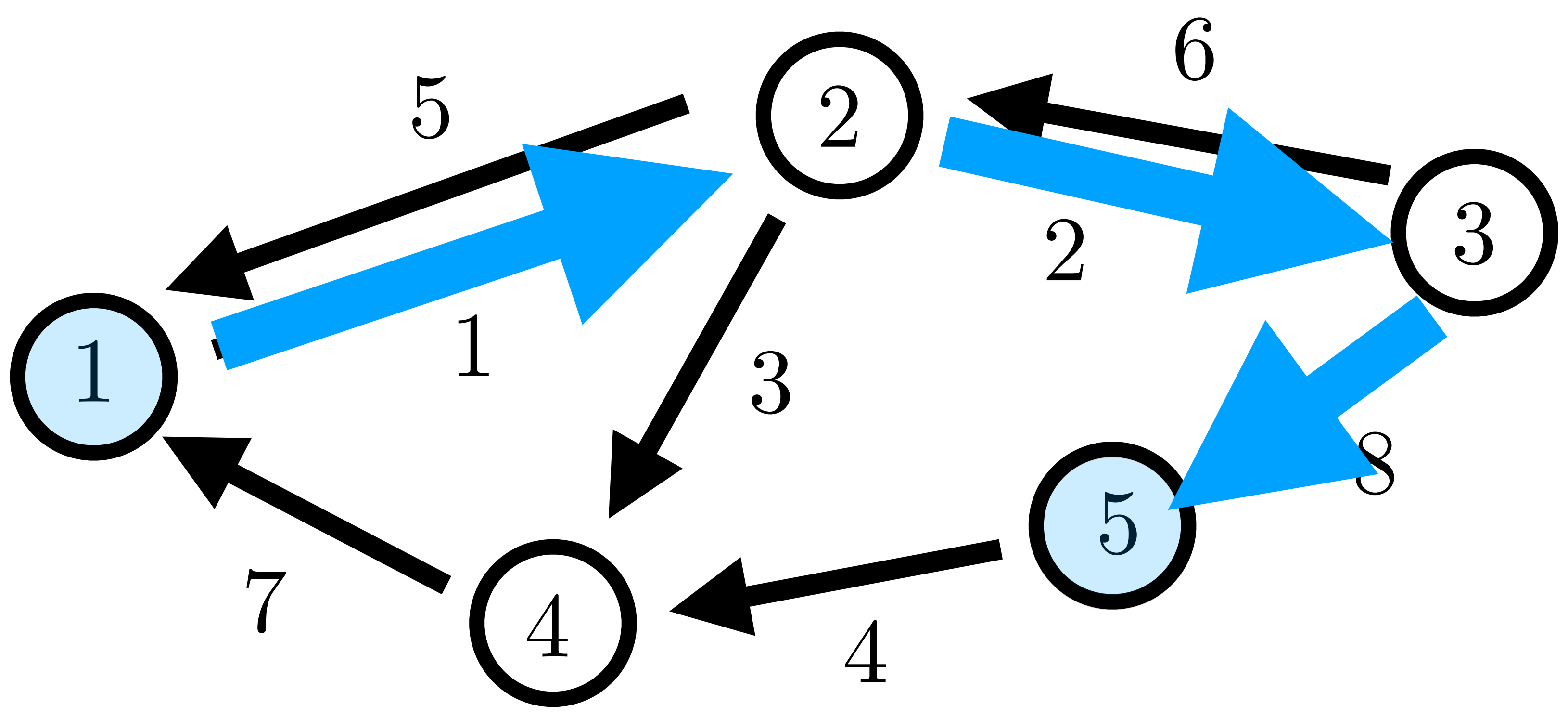

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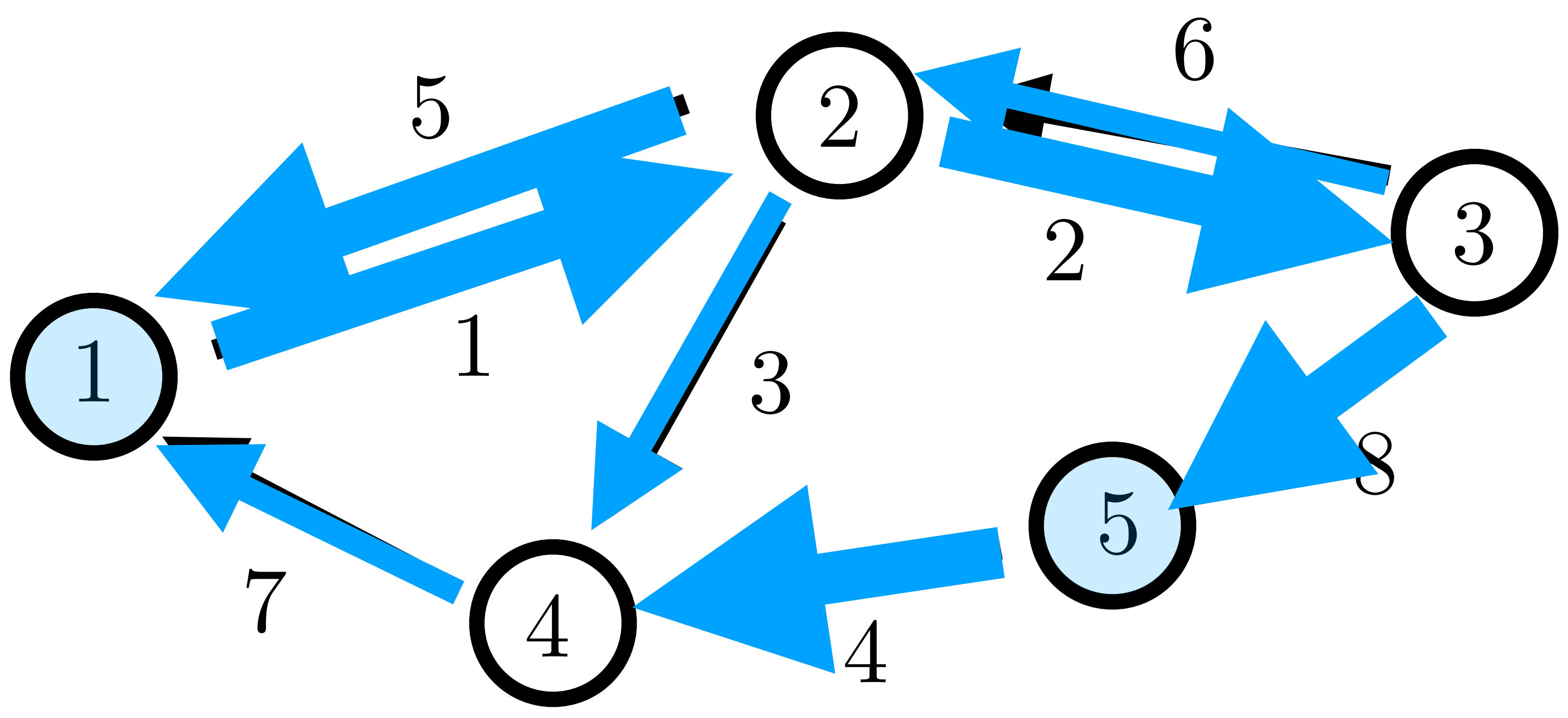
Specific Solution

Cyclic Flow

$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Incidence Matrix - Domain



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Non-conserved flow

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Edge flow vector

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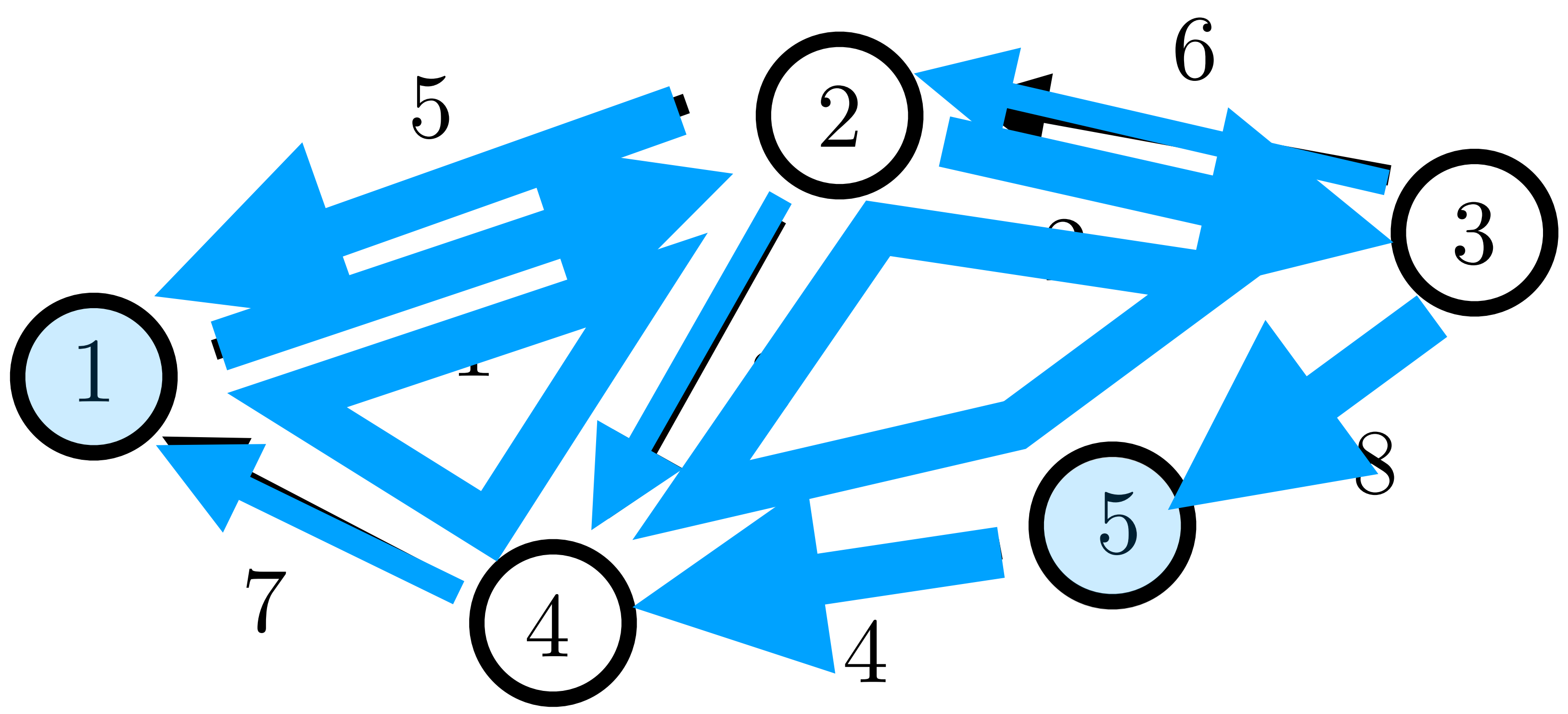
Specific Solution

Cyclic Flow

Incidence Matrix - Domain



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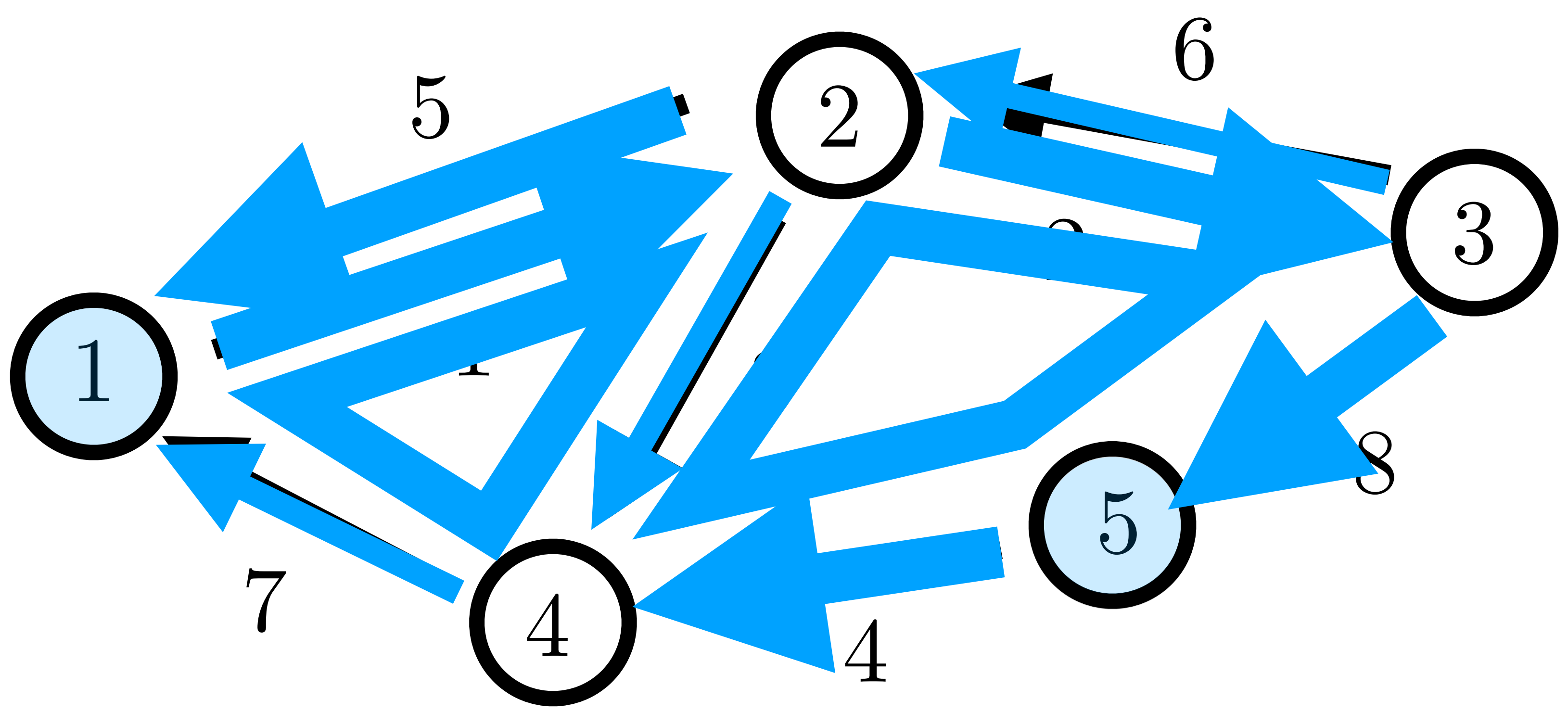
Specific Solution

Cyclic Flow

Incidence Matrix - Domain



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Domain & Co-Domain Interpretation

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Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \underbrace{\bar{x}}_{\text{Specific Solution}} + \underbrace{Cz}_{\text{Cyclic Flow}}$$

Specific Solution

Cyclic Flow

Minimum Norm Solution: $x = D^T(DD^T)^\dagger S$

… no component of x in nullspace, ie. no cycle flows

Moore Penrose Pseudoinverse

… gives the minimum norm/least squares solution

… to be an exact solution S needs to be in range of D

(conservation of flow in & out of network)

Incidence Matrix - Co-Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

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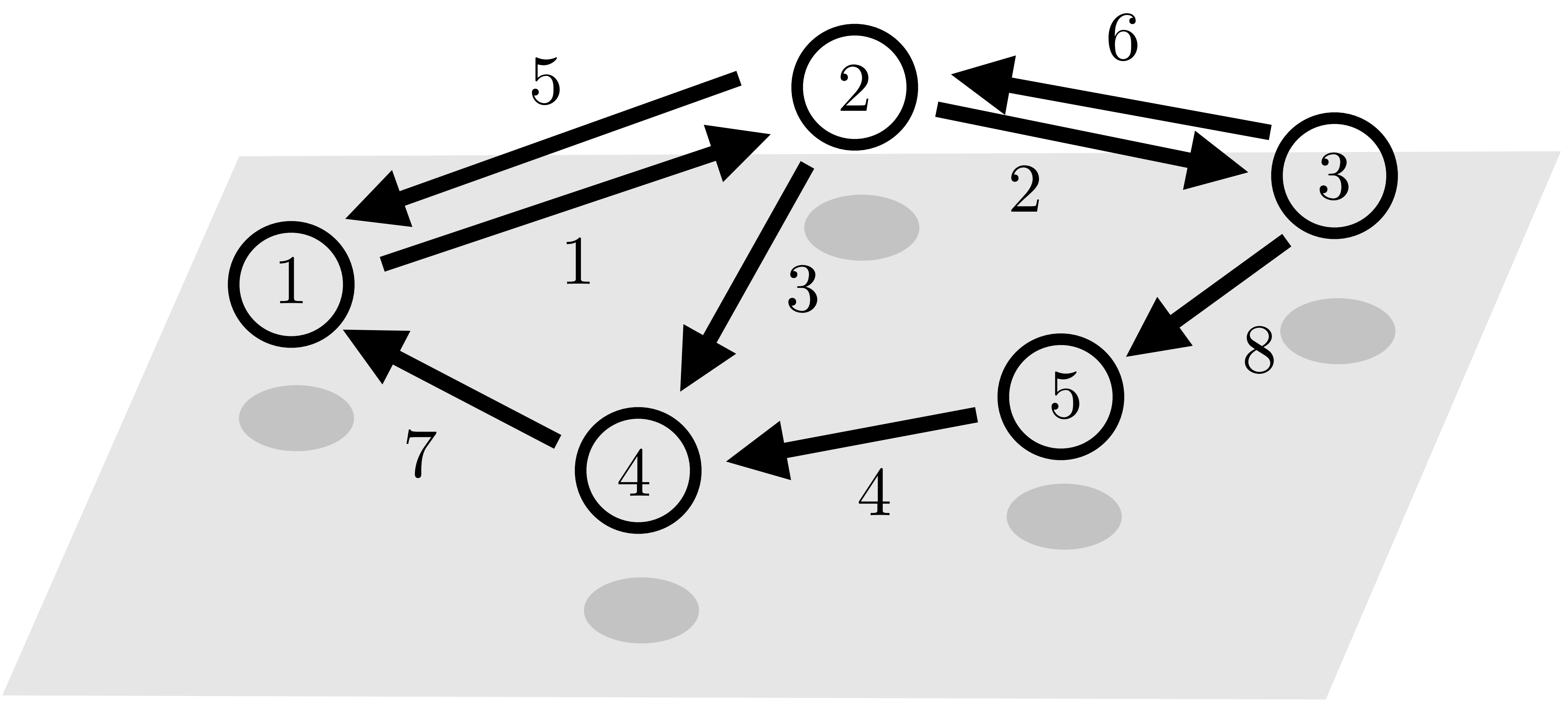

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Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

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Non-conserved flow

$$S = Dx$$

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Specific Solution

Cyclic Flow

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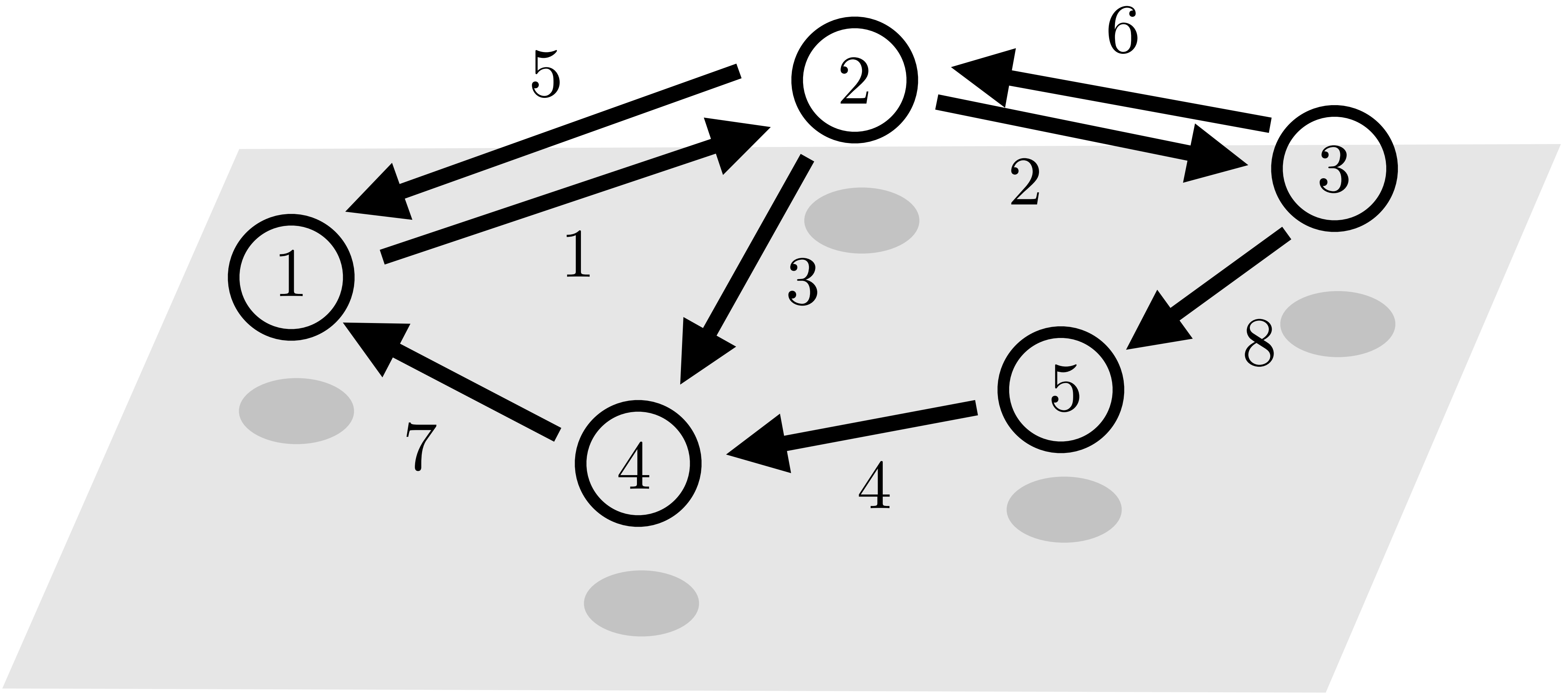

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Non-conserved flow $$S = Dx$$ **Edge flow vector**

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Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

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Examples

...gravitational potential

...voltage

...cost-to-go

Incidence Matrix - Co-Domain

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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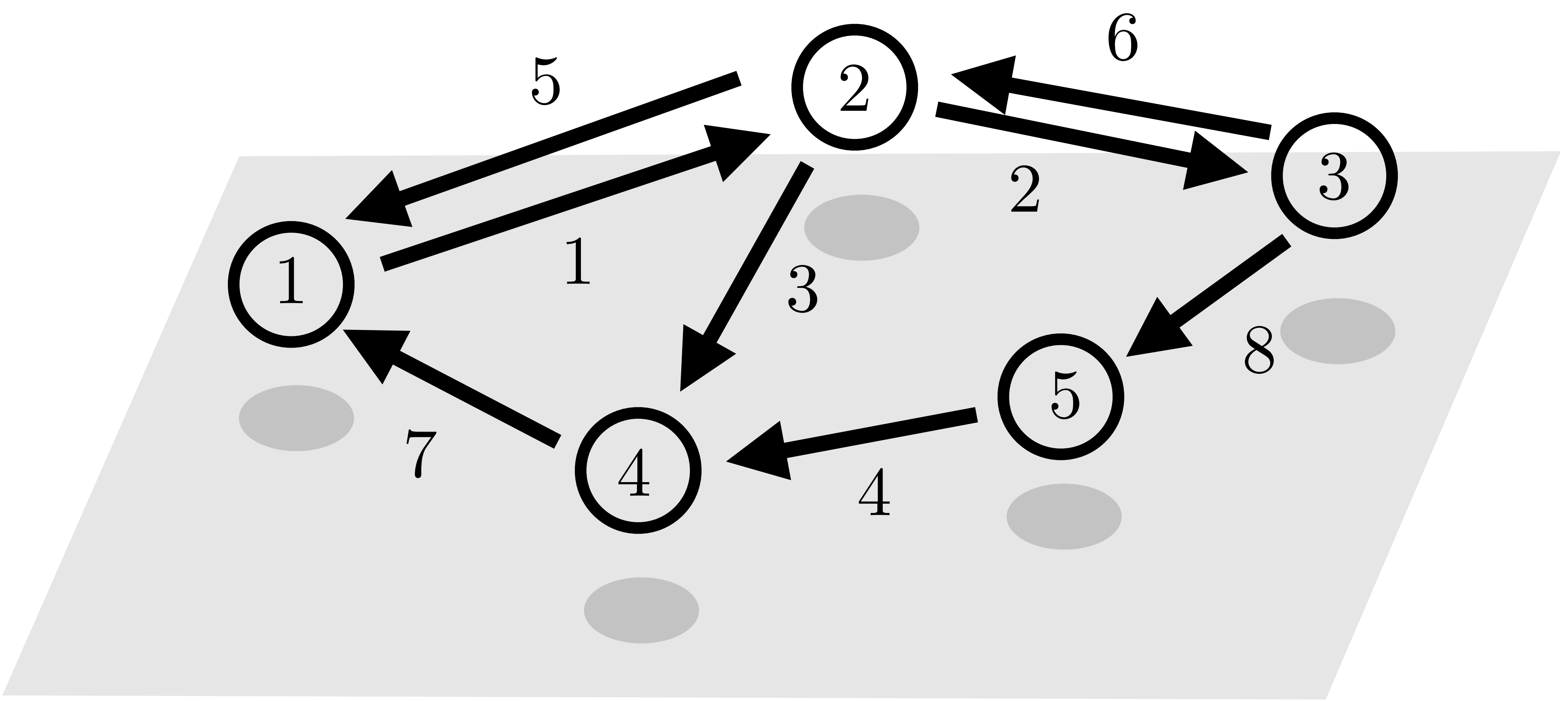

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ **...tension/difference on edges**

Non-conserved flow

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Edge flow vector

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Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

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$$w^T D = \tau^T$$

Value function

Edge tension

Incidence Matrix - Co-Domain

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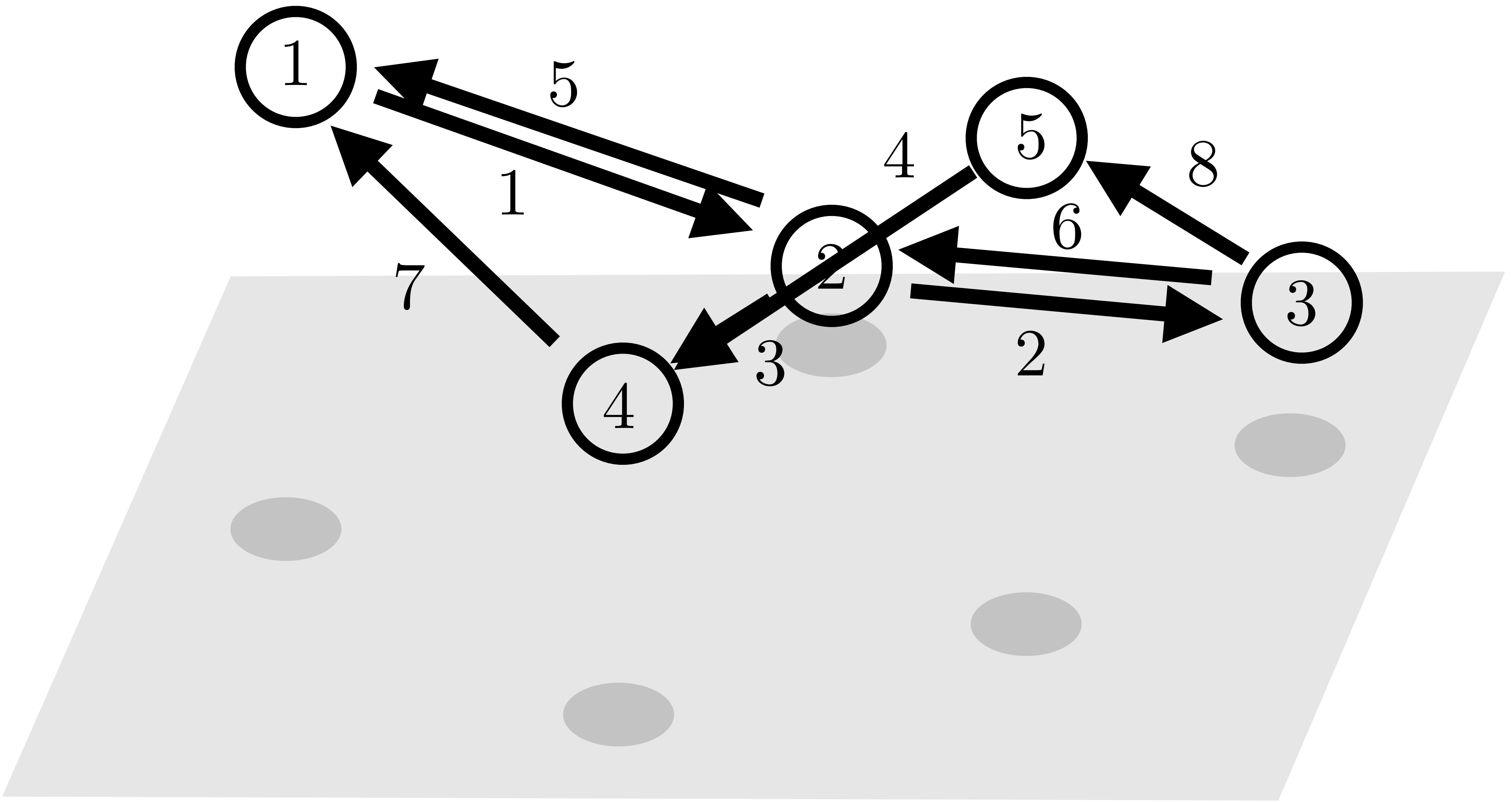

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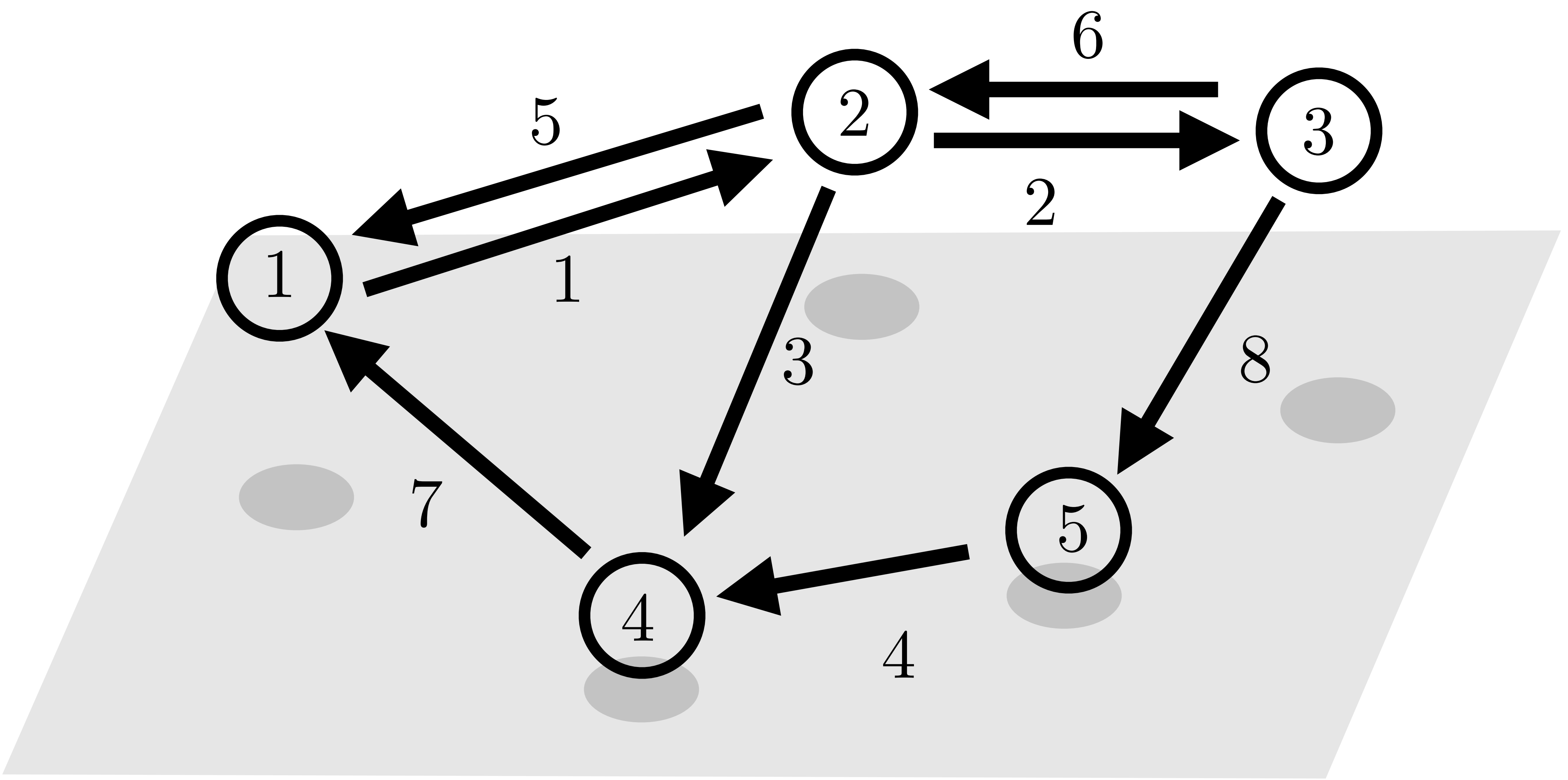
Edge tension

$$\left[w^T D\right]_e = w_i - w_{i'} = \tau_e$$

Incidence Matrix - Co-Domain



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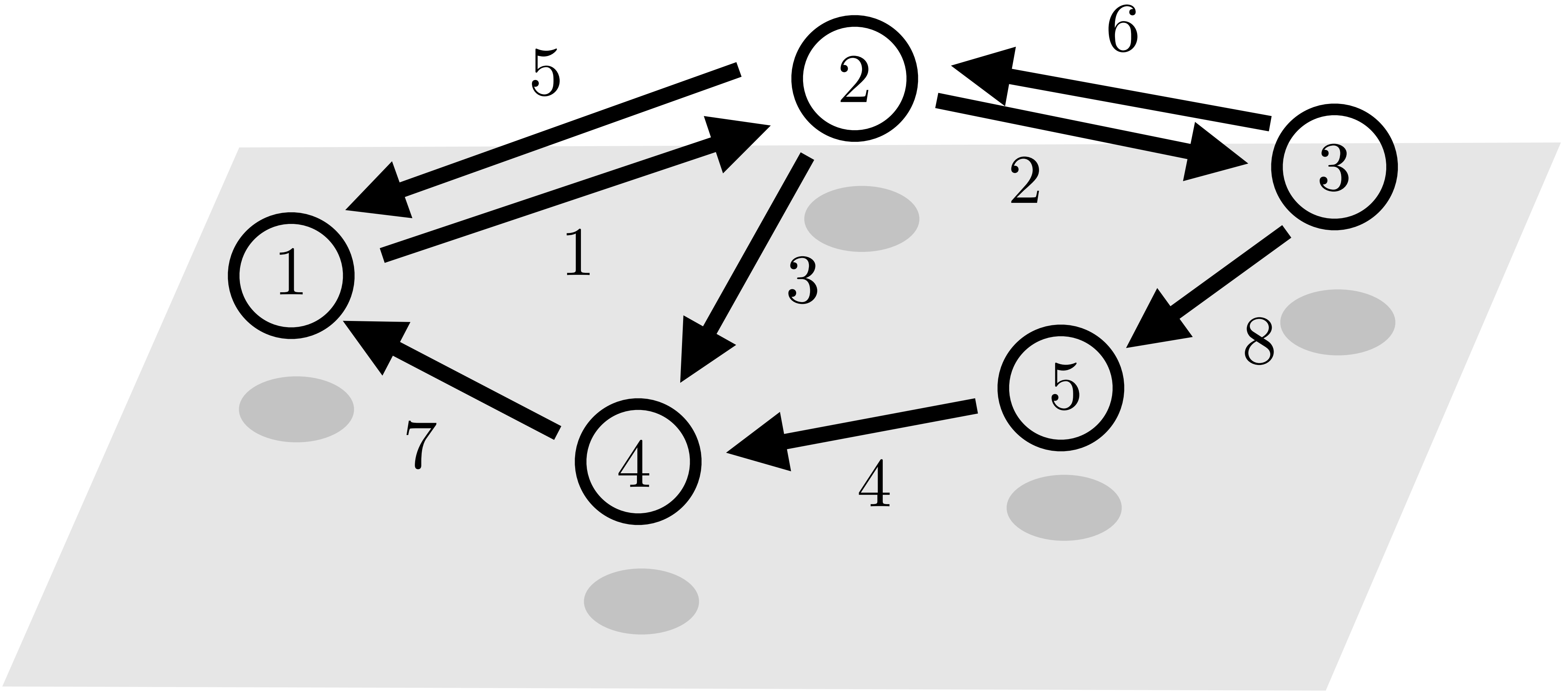

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Incidence Matrix - Co-Domain

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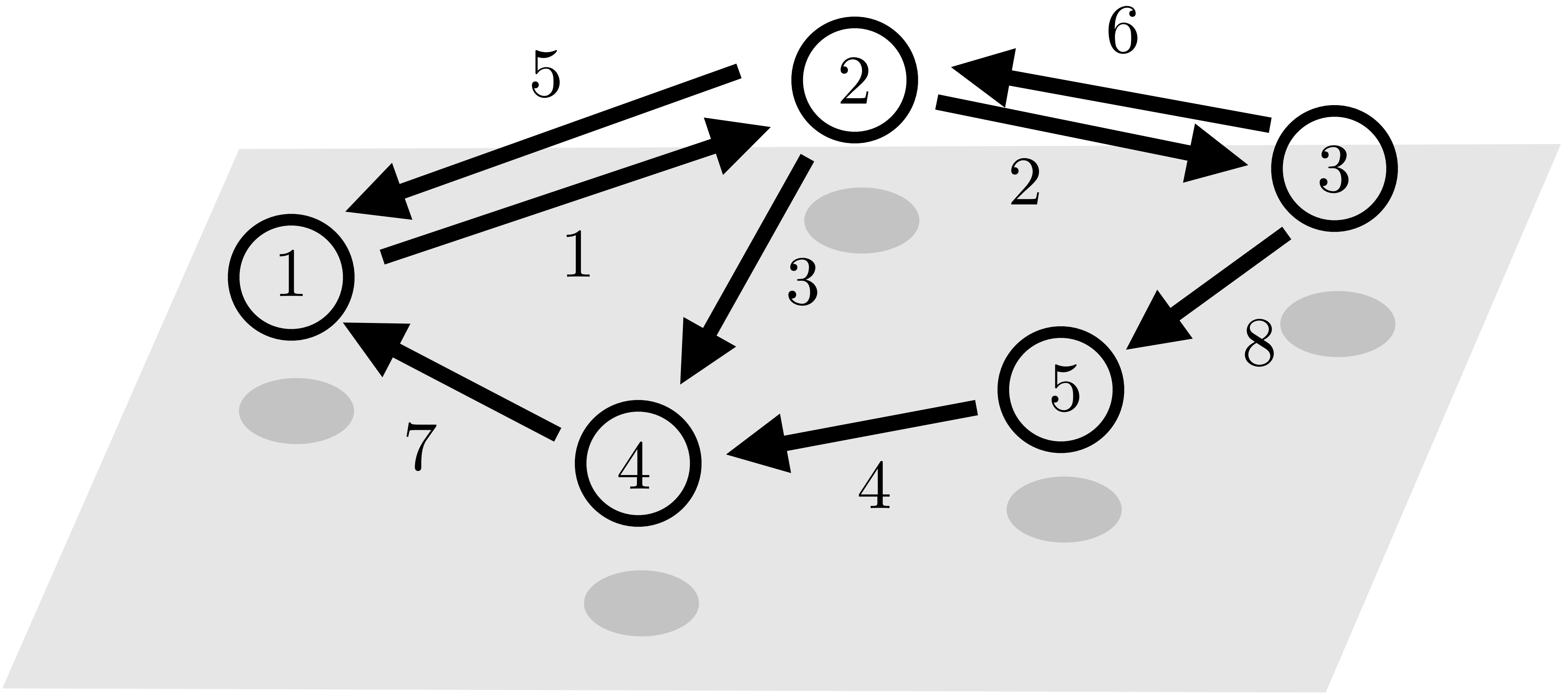

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Cyclic Flow

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$$w^T D = \tau^T$$

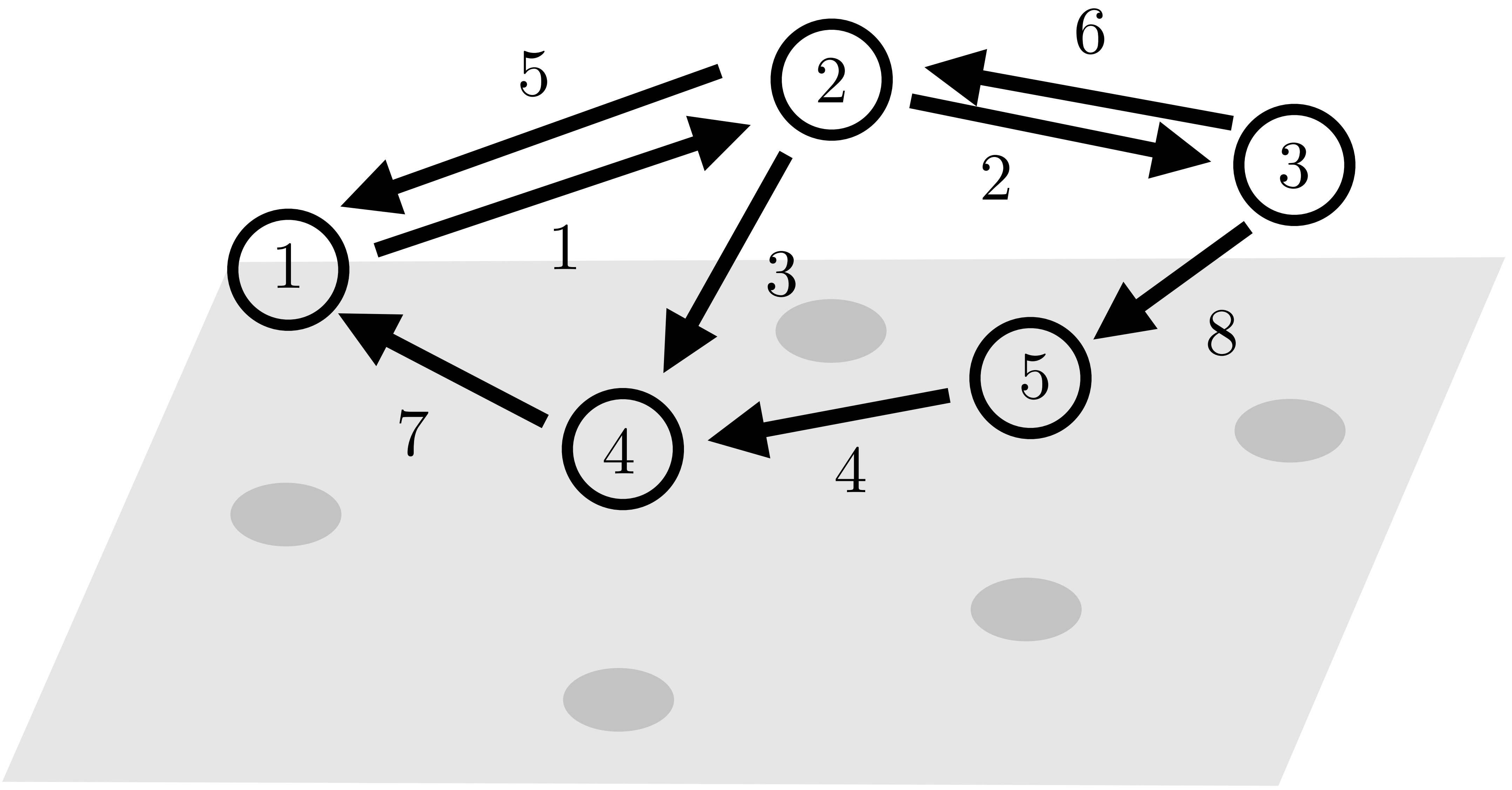
Value function

Edge tension

$$(w^T + \mathbf{1}^T)D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Co-Domain



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Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

$\tau \in \mathbb{R}^{|\mathcal{E}|}$ ...tension/difference on edges

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $S \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

$w \in \mathbb{R}^{|\mathcal{V}|}$ ...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

$$(w^T + \mathbf{1}^T)D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Co-Domain

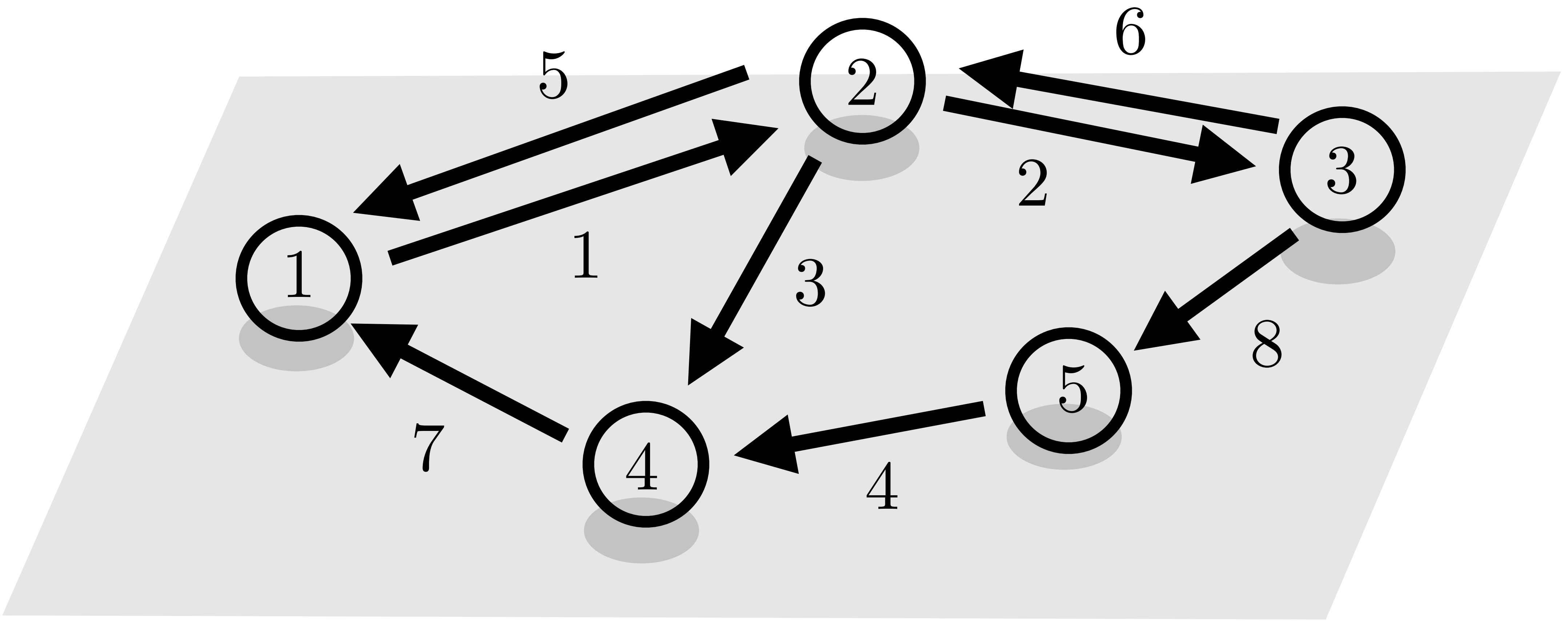
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\qquad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



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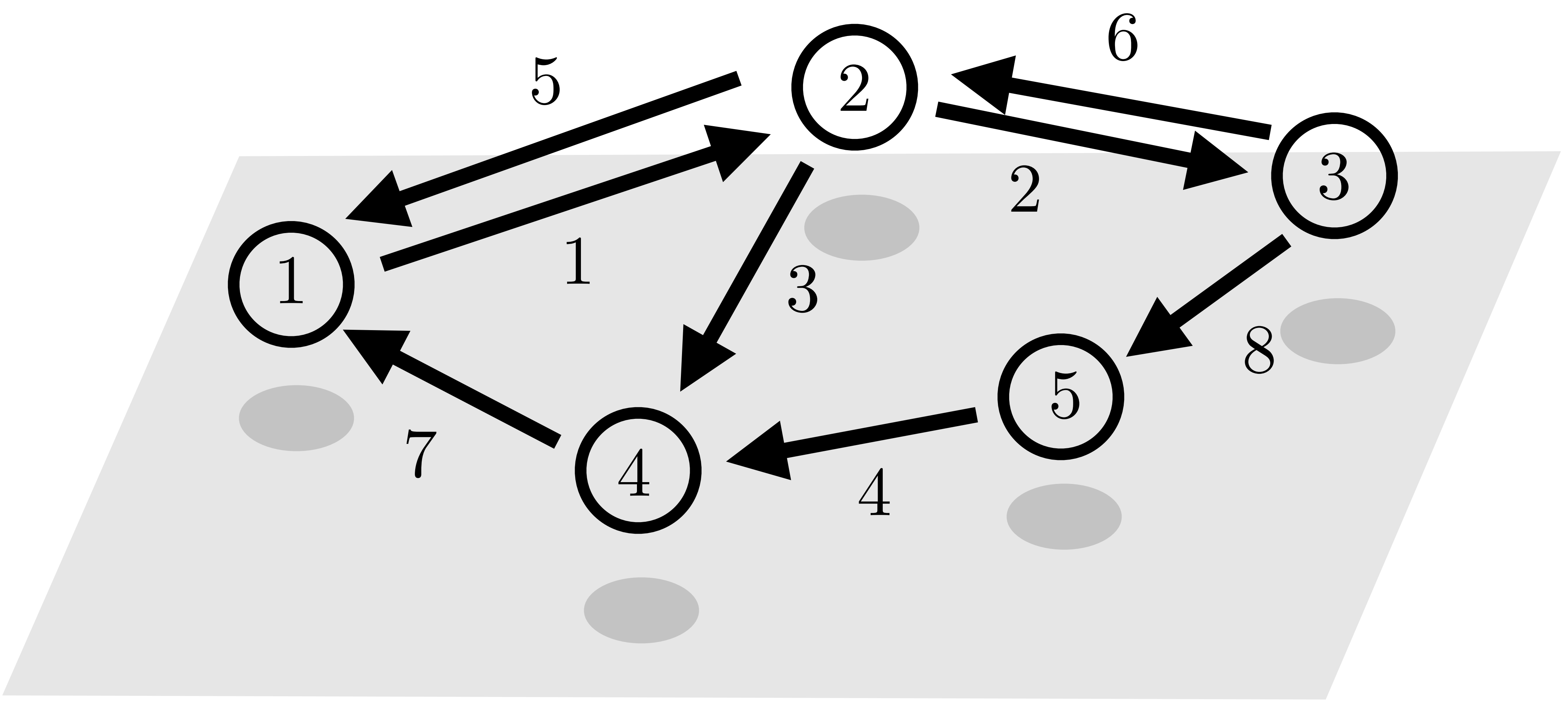

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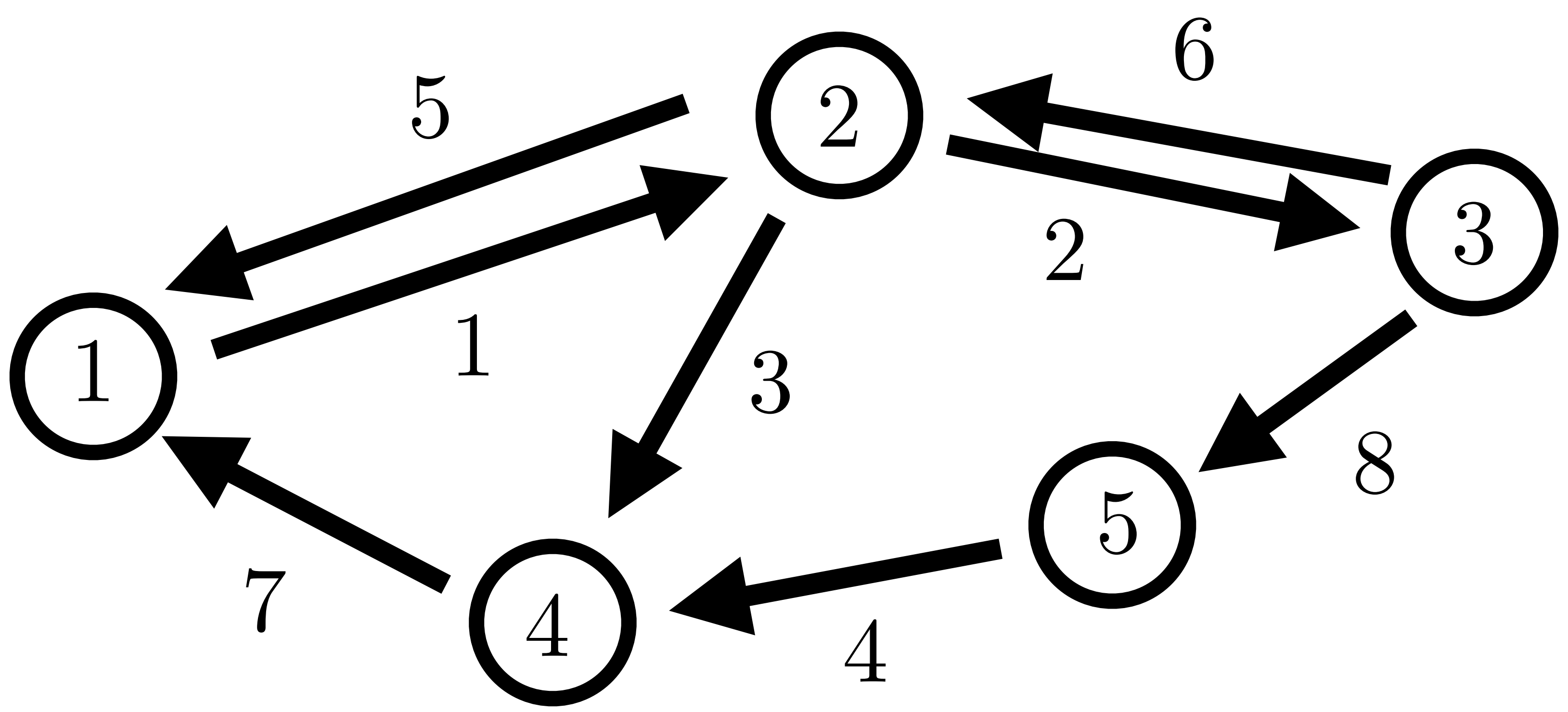


Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fundamental Thm of Linear Algebra

$$A \in \mathbb{R}^{m \times n}$$

rank A = k

$$\begin{bmatrix} \; A \; \end{bmatrix}$$

Rank-nullity

$$\text{rank}(A) + \text{null}(A) = n$$

Co-Domain

Range A

dim = k

$\oplus^{\perp}$

Nullspace A^T

dim = m-k

"Span of the columns"

"Orthogonal to columns"

Domain

Range A^T

dim = k

$\oplus^{\perp}$

Nullspace A

dim = n-k

"Span of the rows"

"Orthogonal to rows"

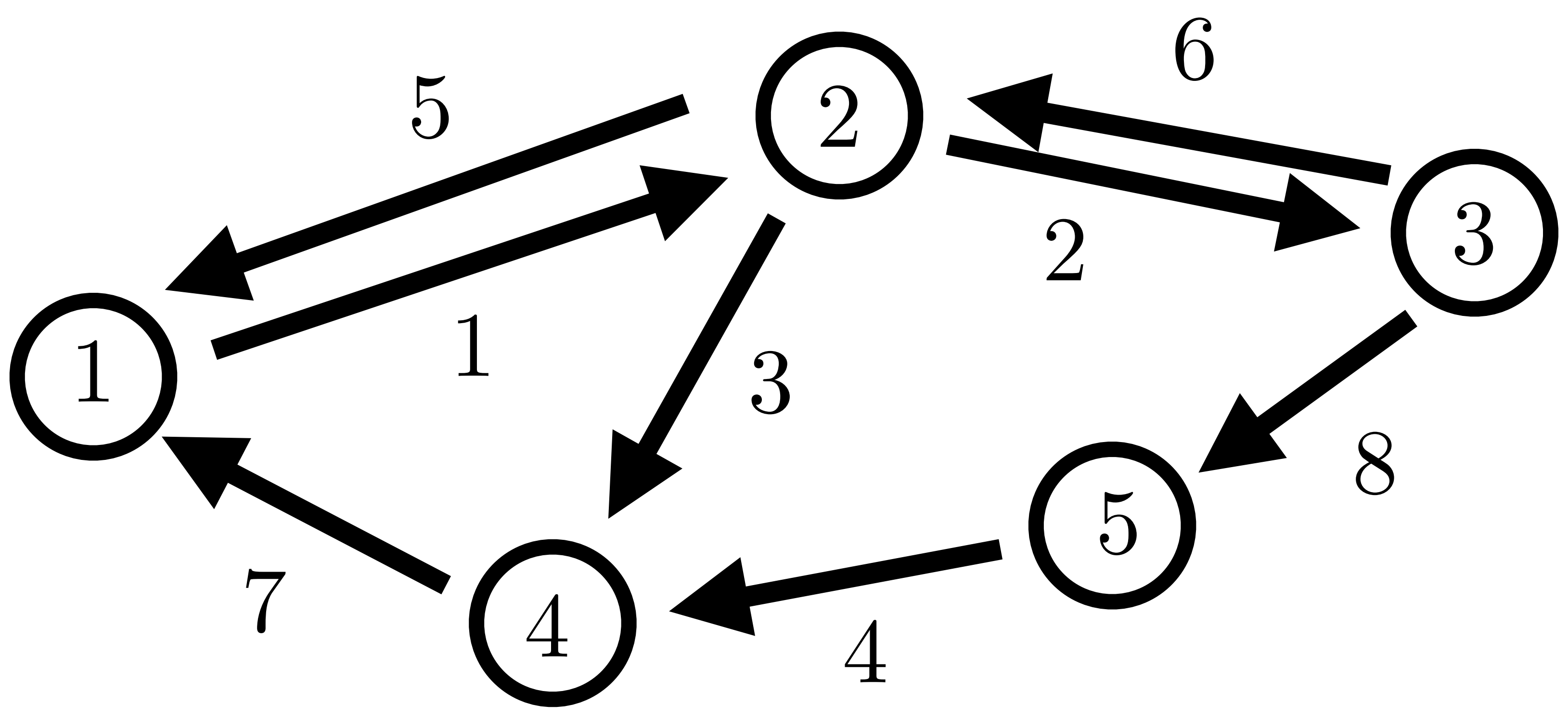
Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

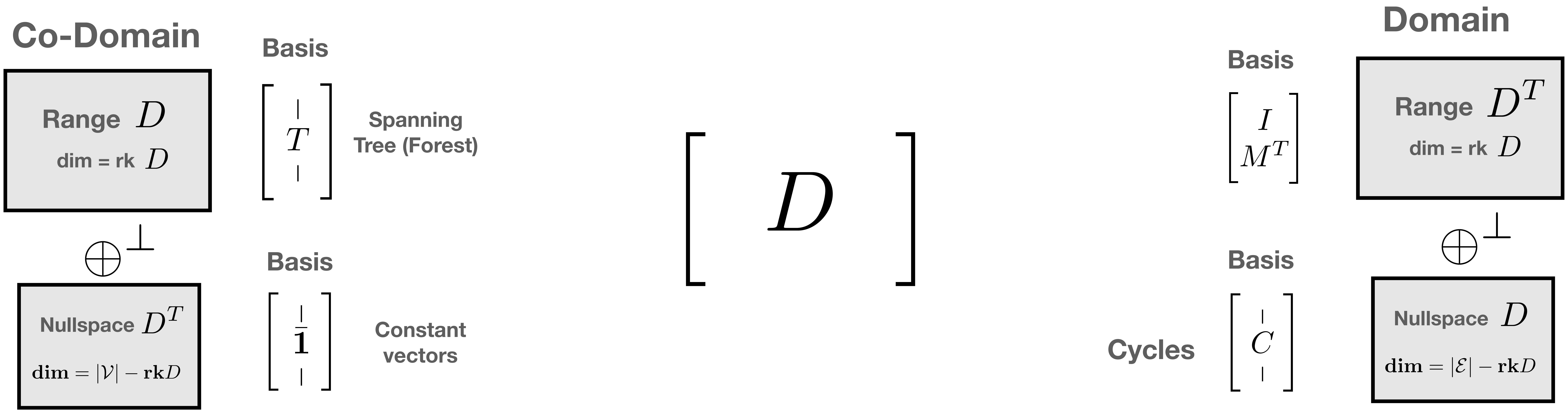
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$



$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Structure Overview



Co-Domain

Range D

dim = rk D

$\bigoplus^{\perp}$

Nullspace D^T

$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\begin{bmatrix} \quad D \quad \end{bmatrix}$$

Domain

Range D^T

dim = rk D

$\bigoplus^{\perp}$

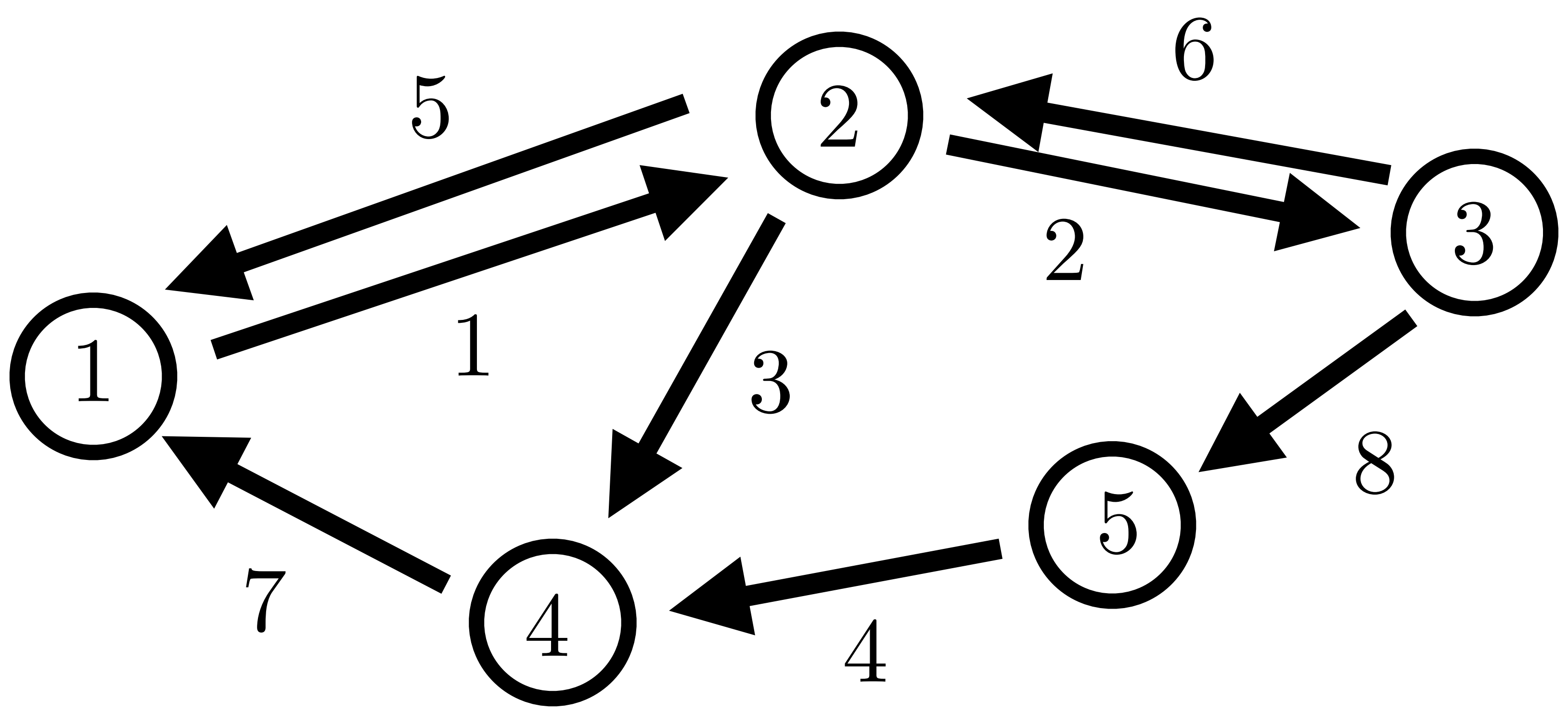
Nullspace D

$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Basis Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Incidence Matrix



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Matrix Structure Overview

Co-Domain

Range D

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Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ **Constant vectors**

$$\begin{bmatrix} \quad D \quad \end{bmatrix}$$

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T

dim = rk D

$$\bigoplus^{\perp}$$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D

$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

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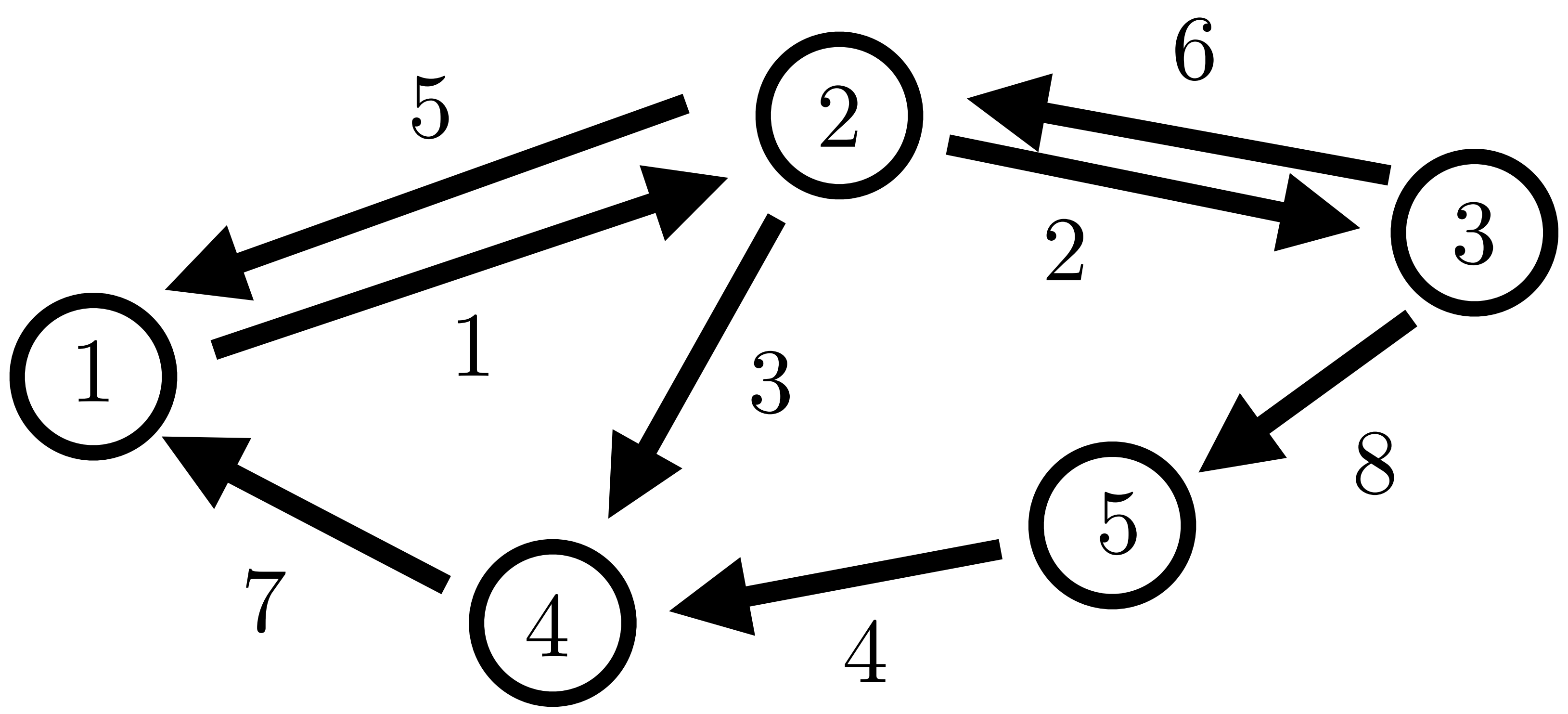

Matrix Structure Overview

Co-Domain

Range D
dim = rk D

$\bigoplus^{\perp}$

Nullspace D^T
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Basis

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis

$\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

$$\begin{bmatrix} \quad D \quad \end{bmatrix}$$

$$D = \begin{bmatrix} | & | \\ T & \mathbf{1} \\ | & | \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & M \\ - \; C^T \; - \end{bmatrix}$$

Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = rk D

$\bigoplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
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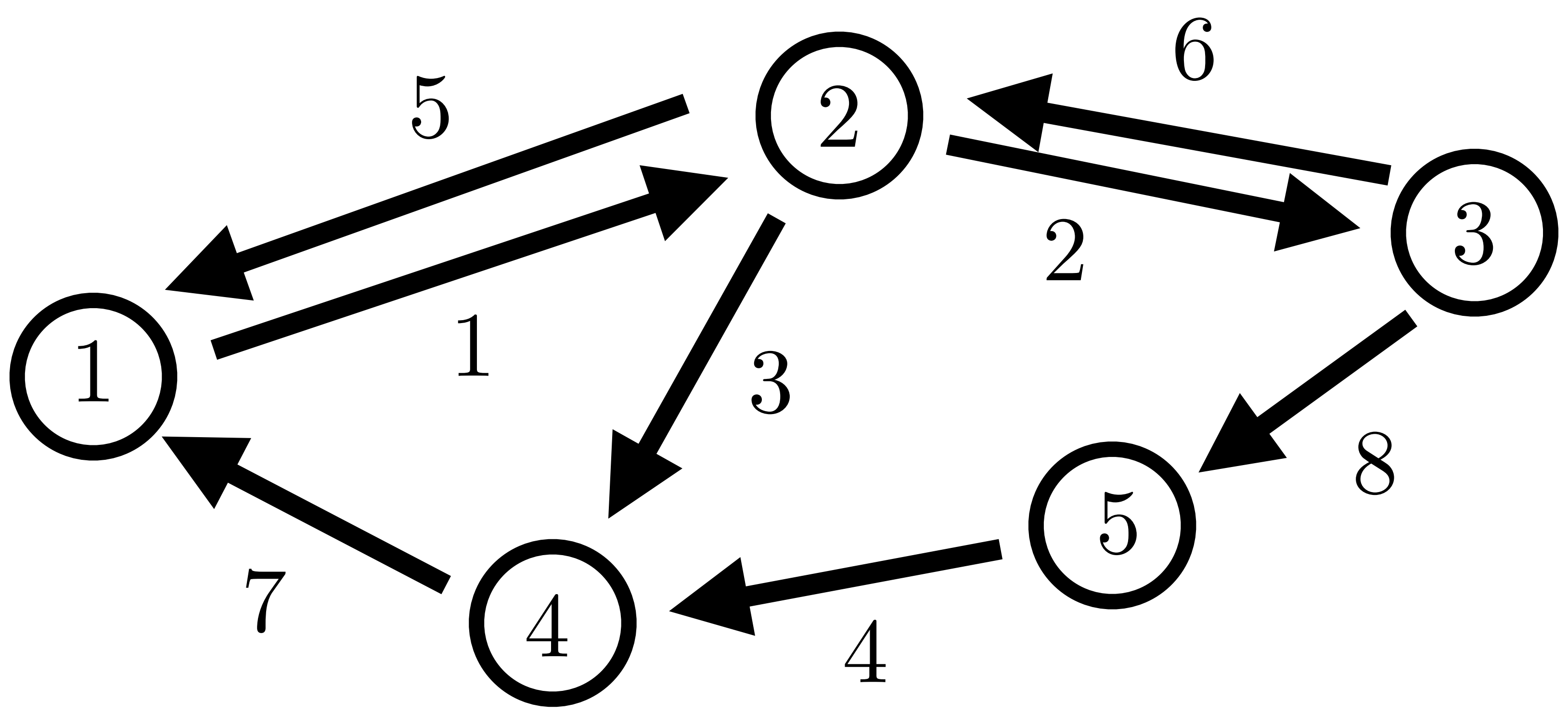
Incidence Matrix



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Matrix Structure Overview

Co-Domain

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Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T

dim = rk D

$\bigoplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D

$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk} D$

Incidence Matrix

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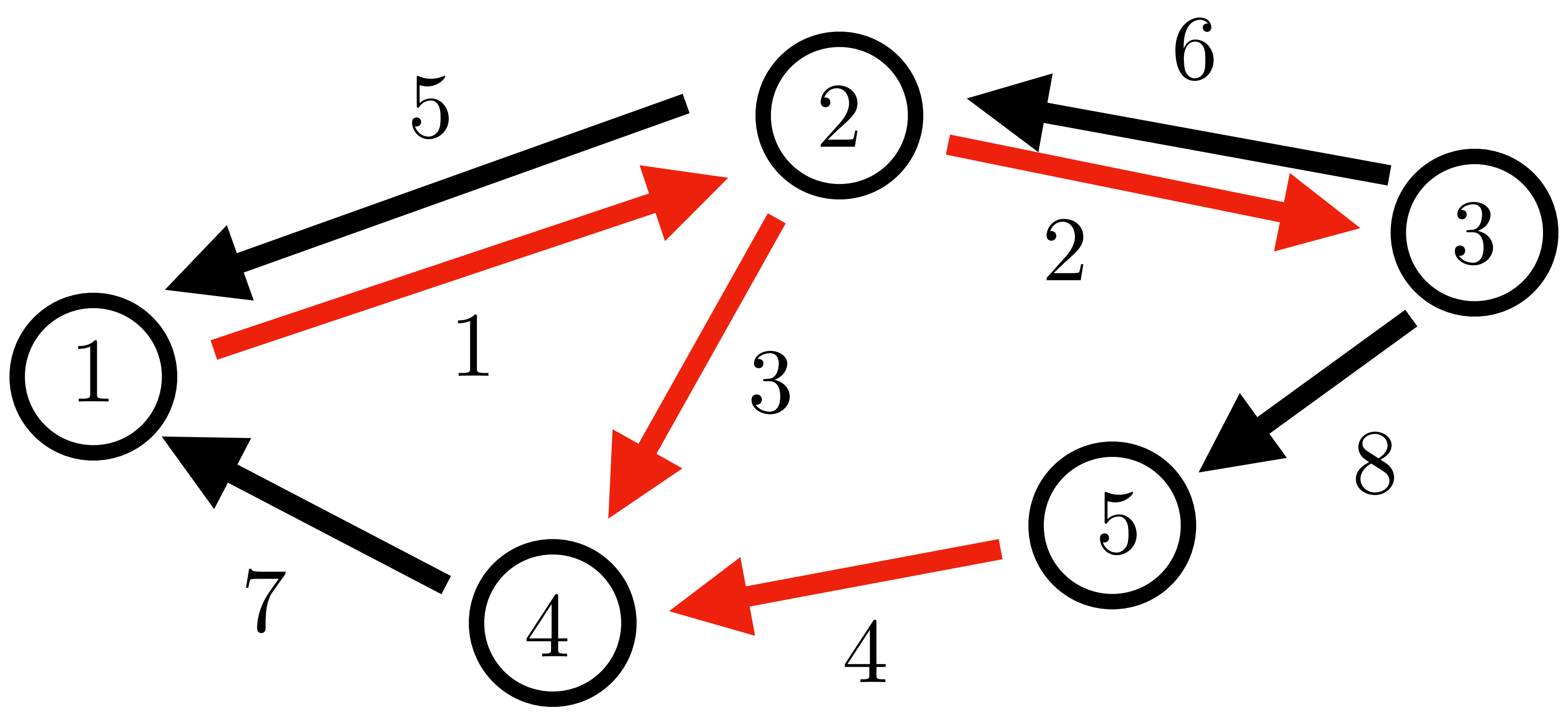

Spanning Tree Construction

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\text{dim} = |\mathcal{V}| - \text{rk}D$

Basis

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis

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Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

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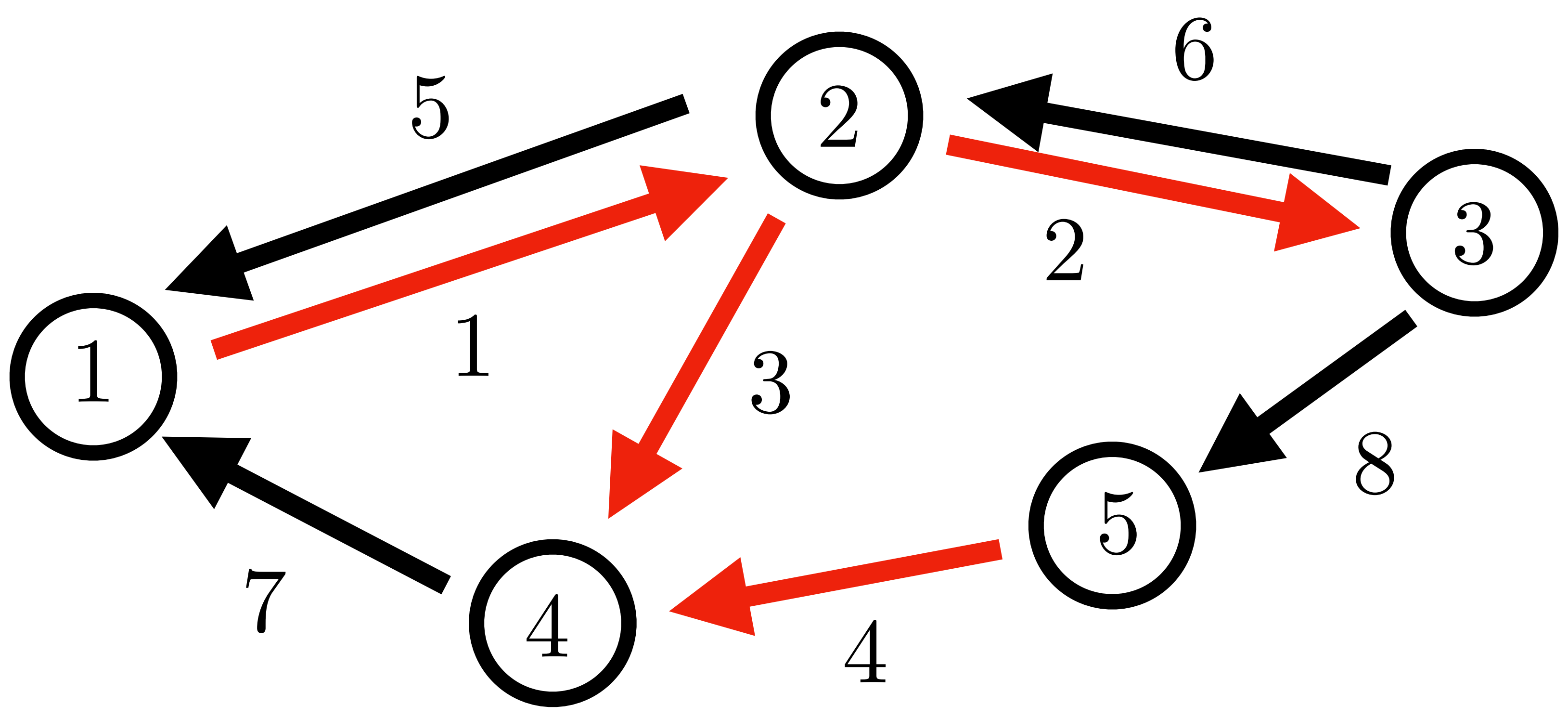

Spanning Tree Construction

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\text{dim} = |\mathcal{V}| - \text{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ Constant vectors

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Spanning Tree (Forest)

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\text{dim} = |\mathcal{E}| - \text{rk}D$

Incidence Matrix

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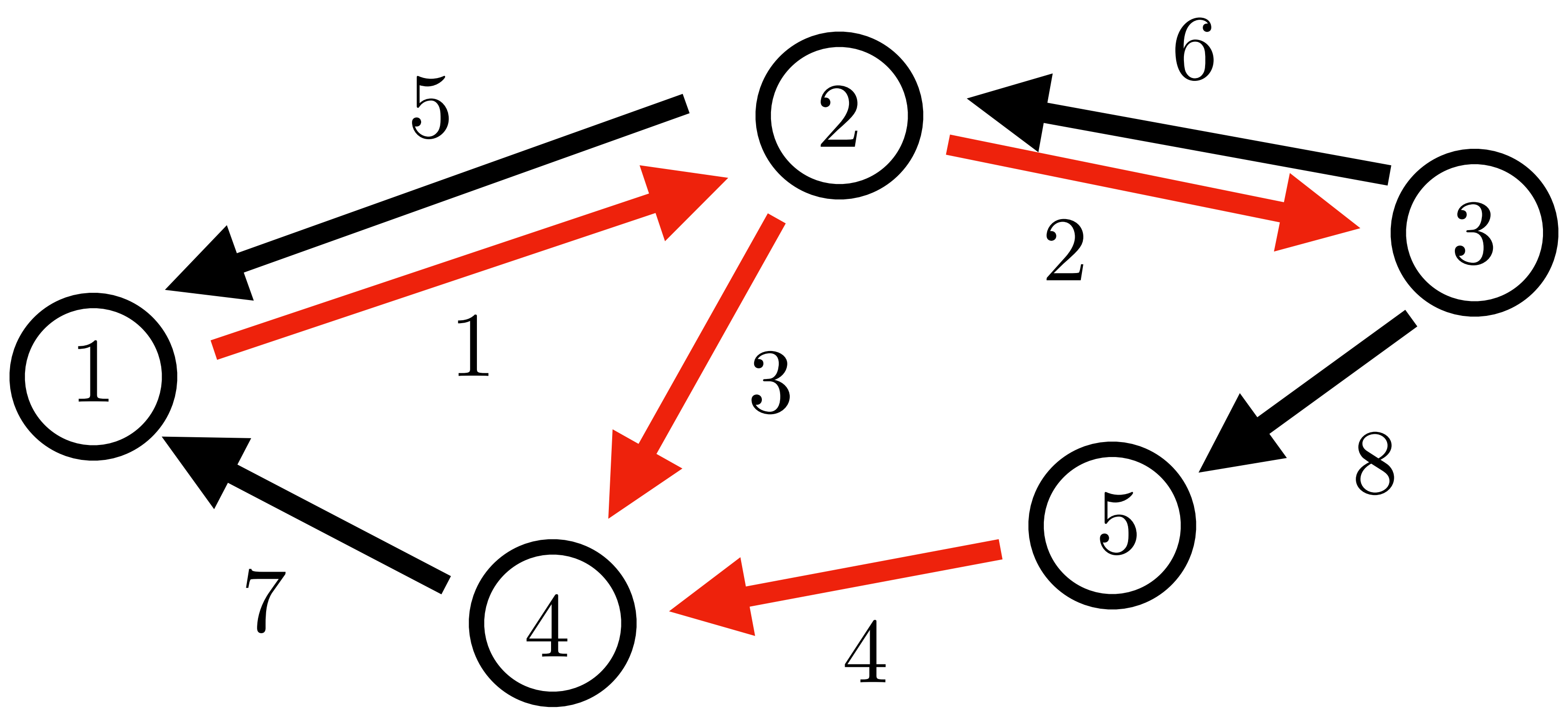

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Spanning Tree Construction

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Spanning Tree (Forest)

Domain

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$\oplus^{\perp}$

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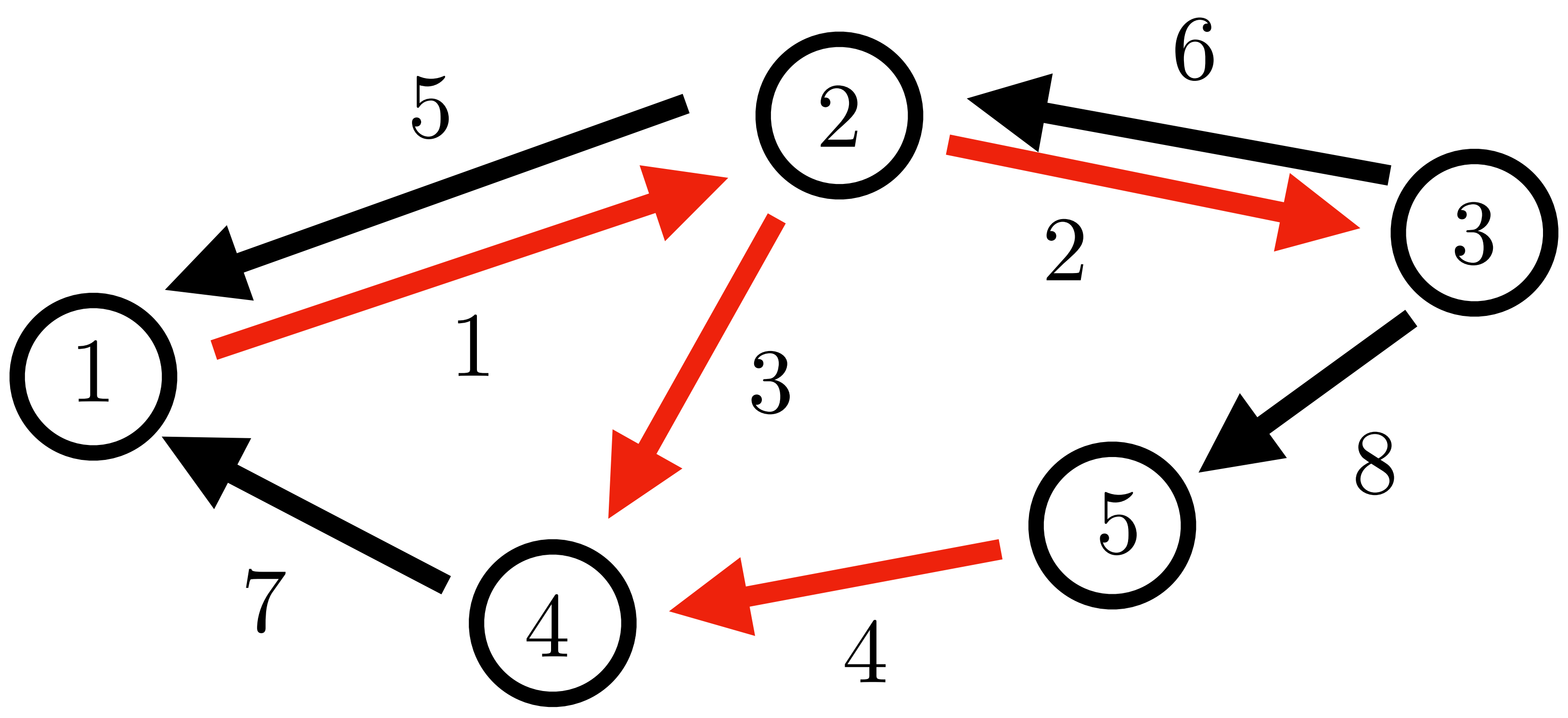

Spanning Tree Construction

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Spanning Tree (Forest)

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\dim = |\mathcal{V}| - \mathrm{rk}D$

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Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis
Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
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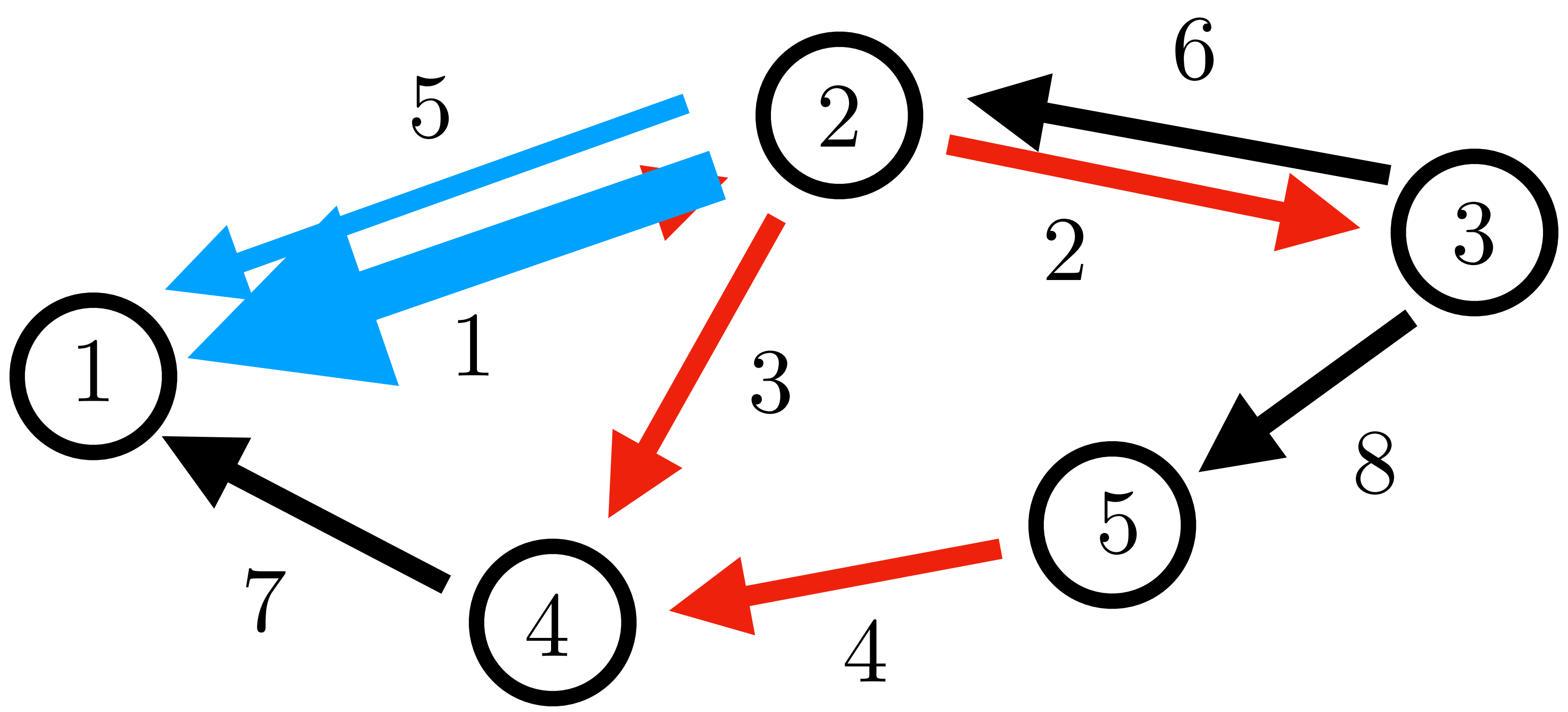

Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

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Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

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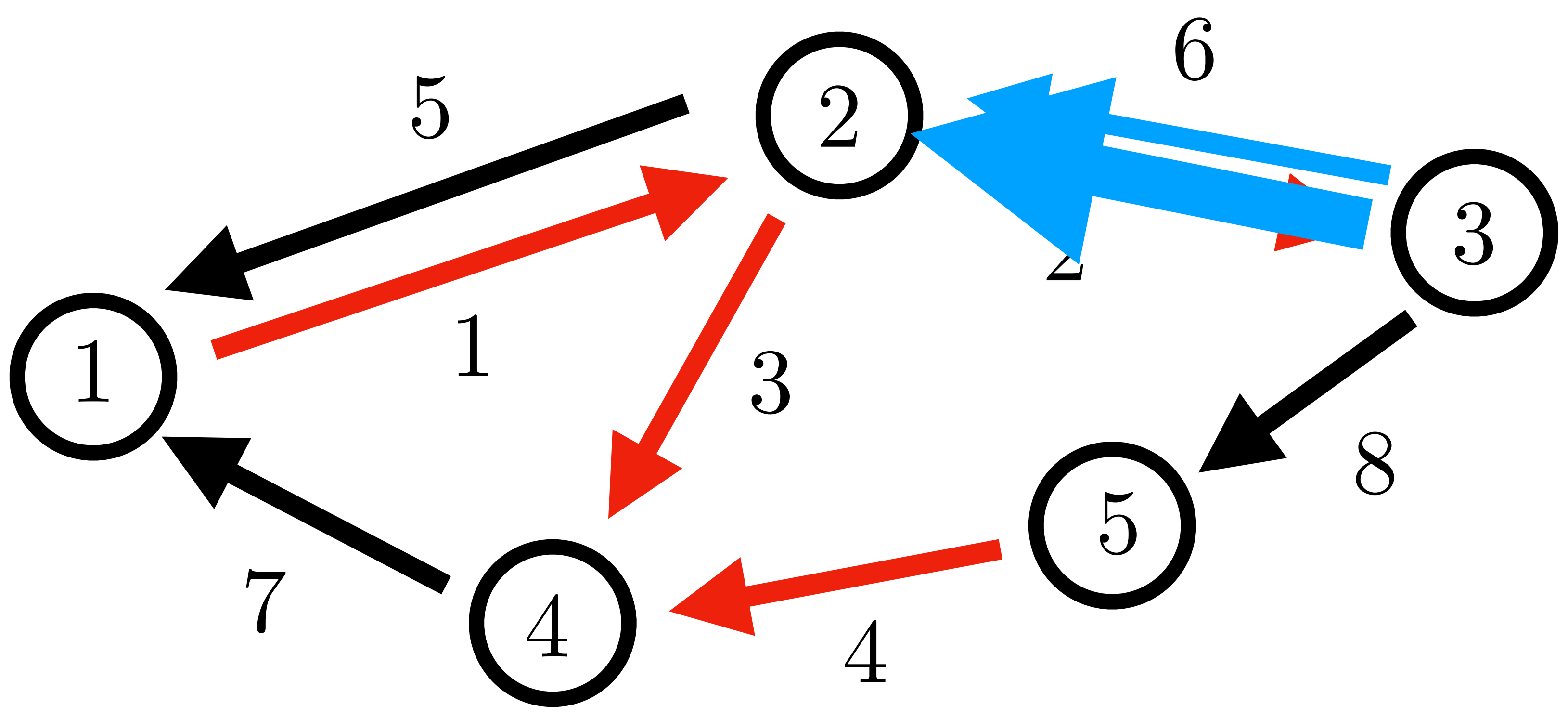

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Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

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Co-Domain

Range D
dim = D

$\oplus \perp$

Nullspace D^T
$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk} D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

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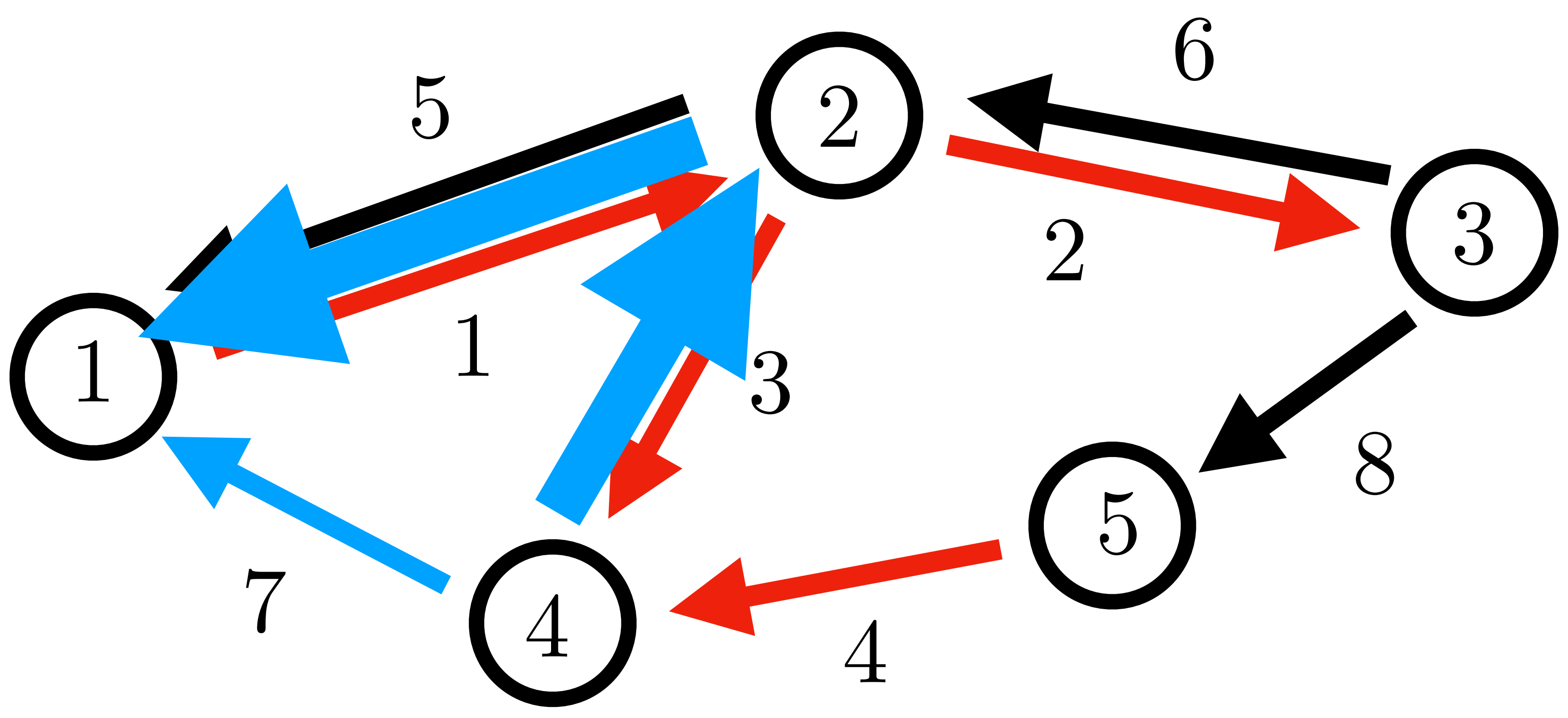
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$



$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

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Nullspace D^T
$\dim = |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus \perp$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

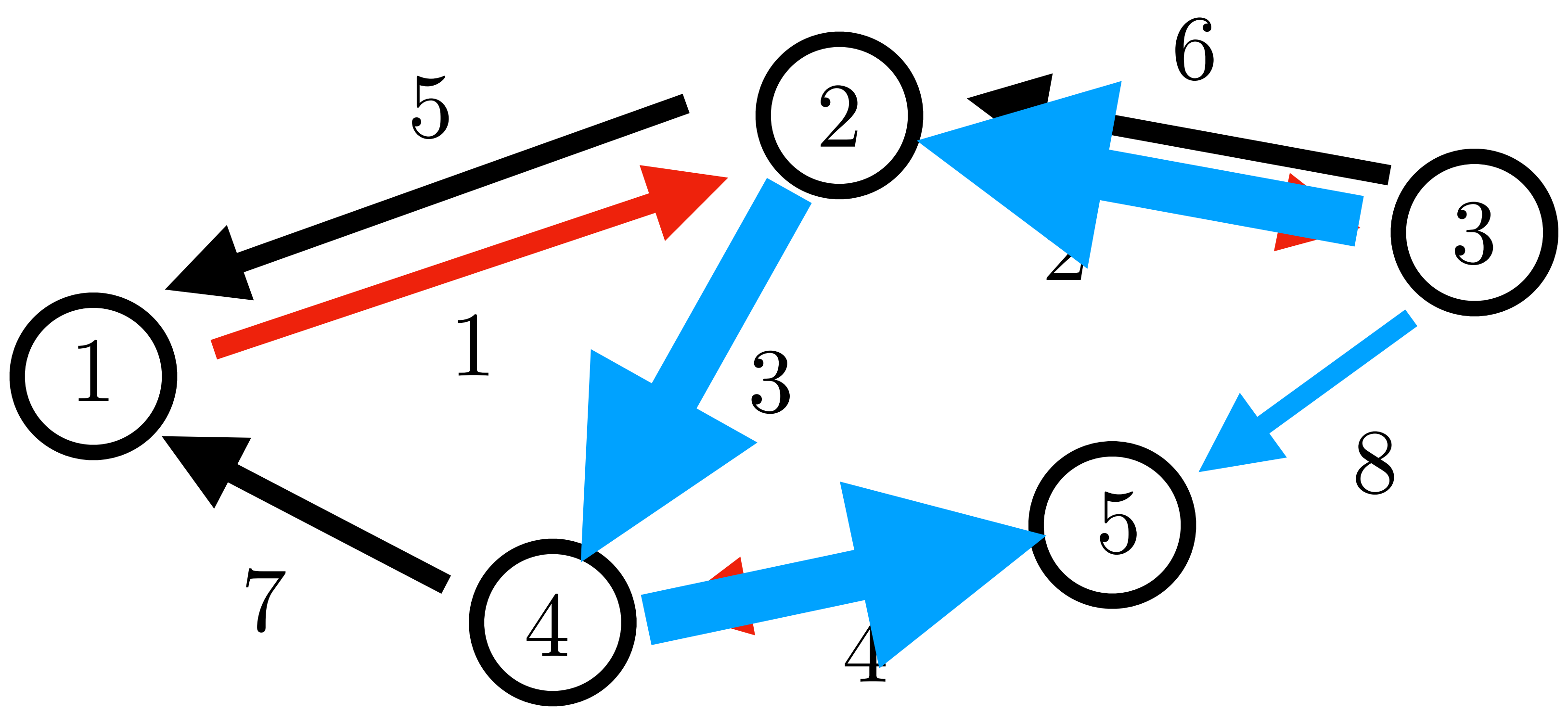
Nullspace D
$\dim = |\mathcal{E}| - \mathrm{rk} D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spanning Tree Construction

$$D = T \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

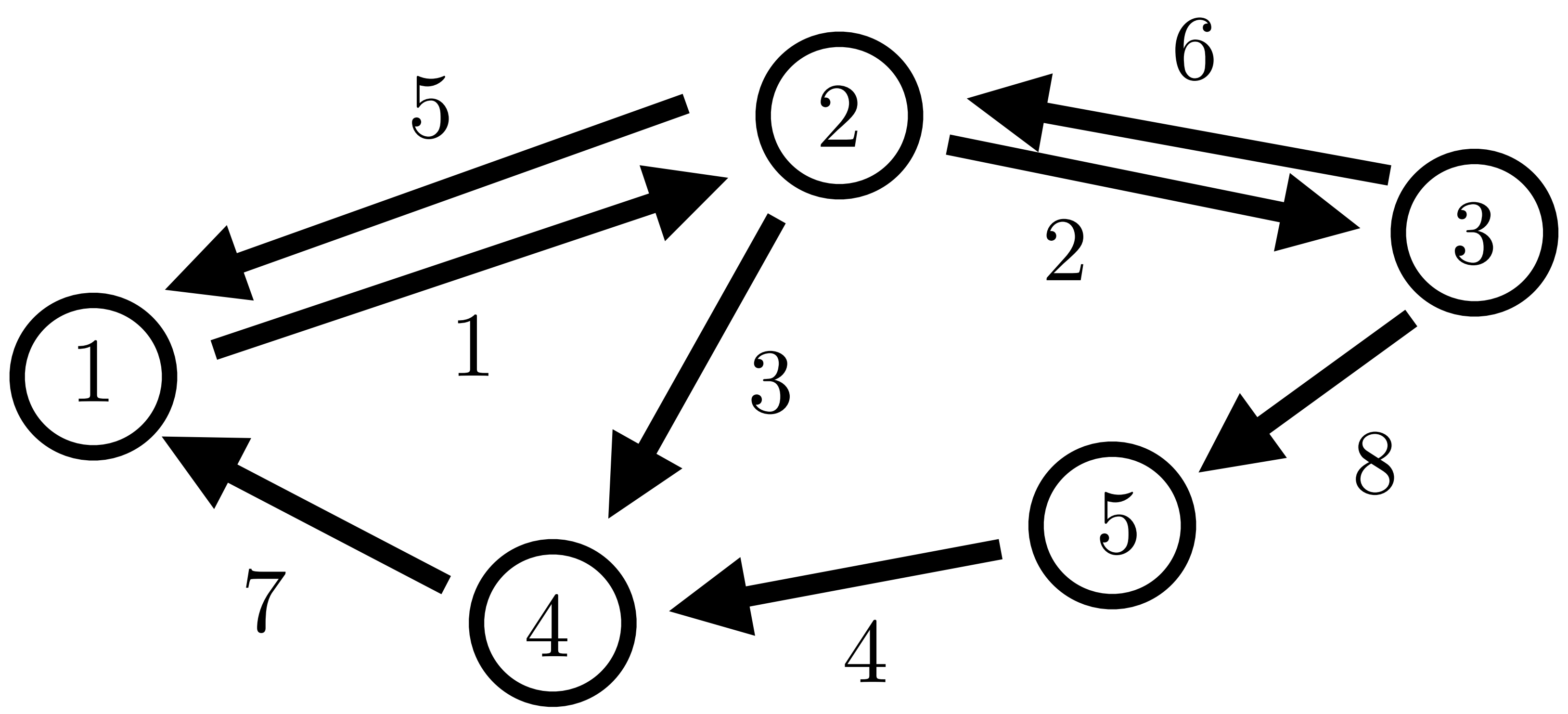
Nullspace D
$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis **Cycles** $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

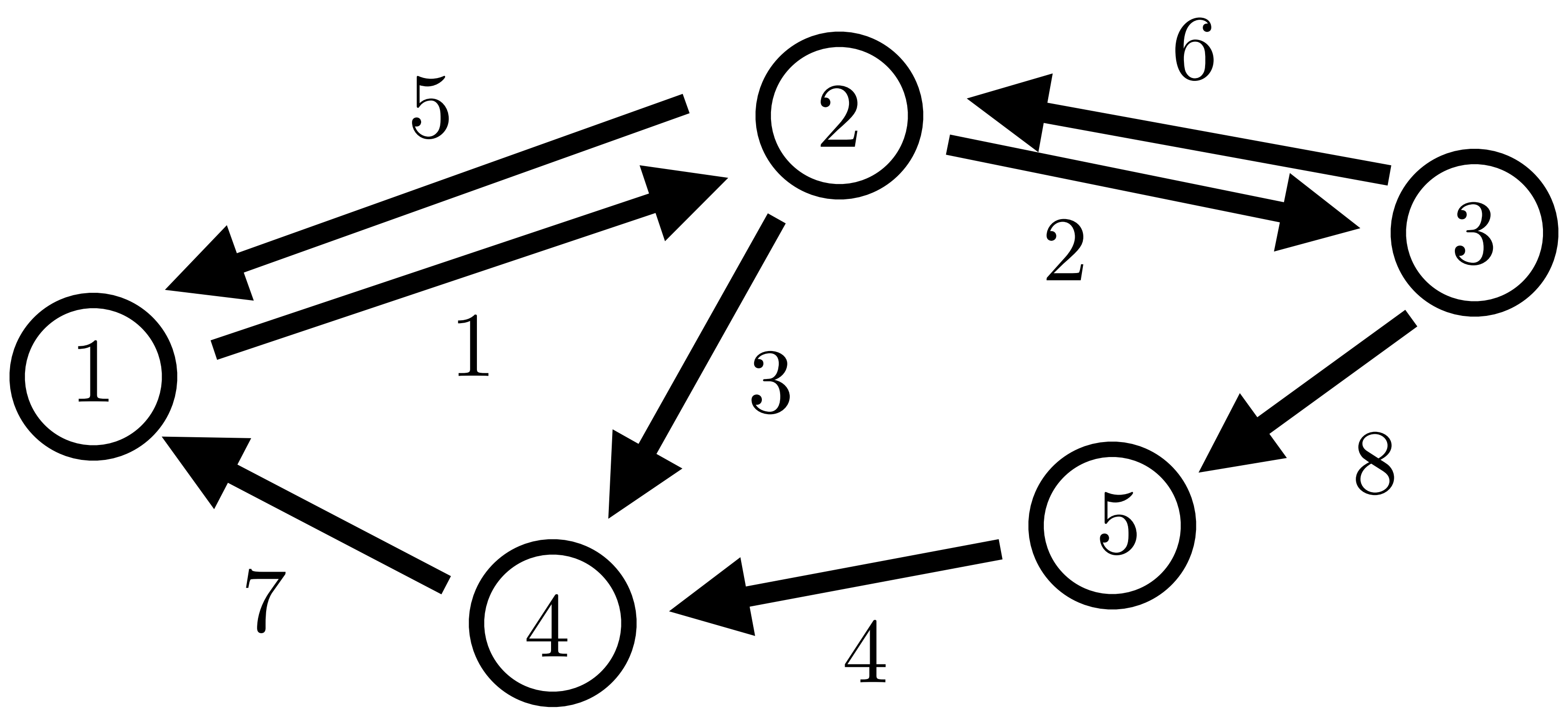
Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\text{dim} = |\mathcal{V}| - \text{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} M \\ -I \end{bmatrix}$$

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
$\text{dim} = |\mathcal{E}| - \text{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

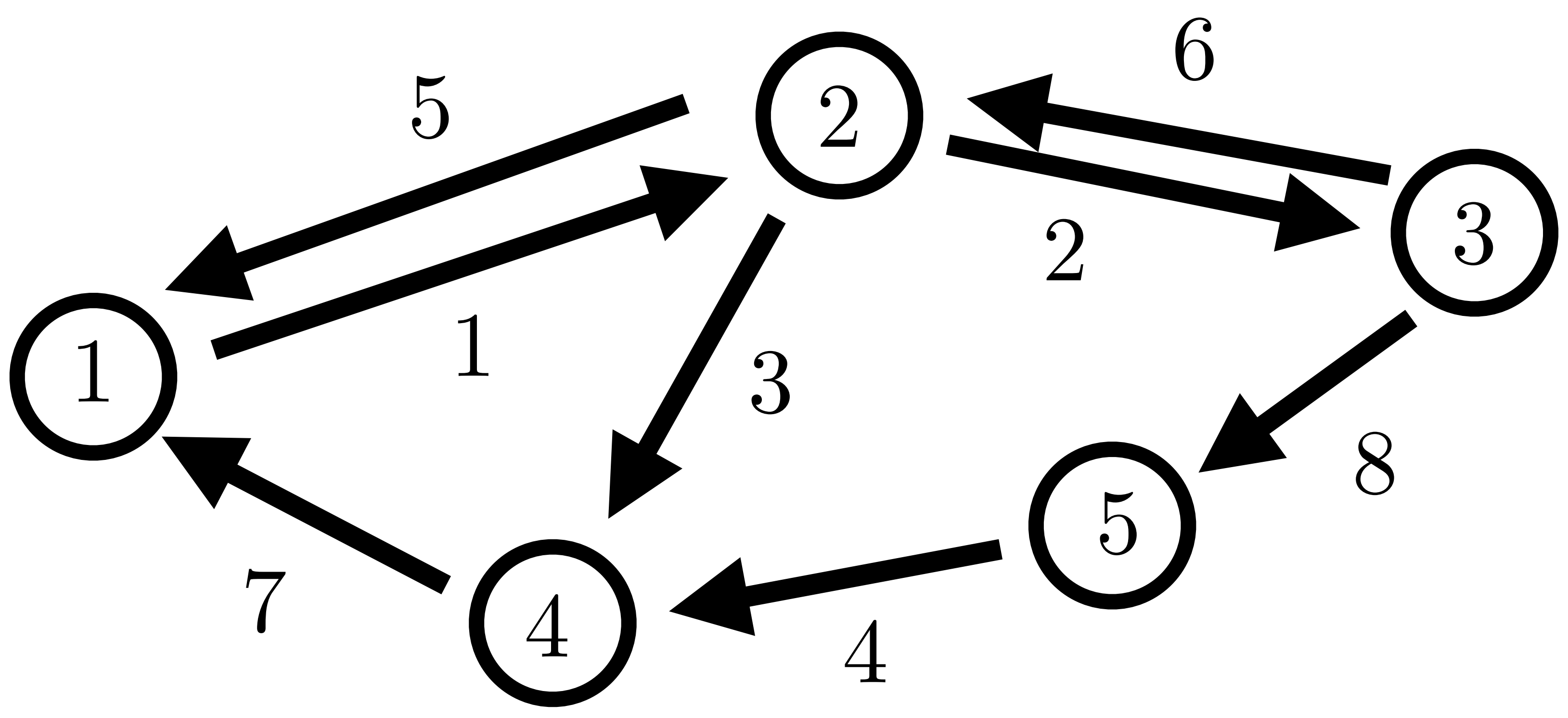
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges



Right Nullspace

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis

$\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ **Constant vectors**

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} M \\ -I \end{bmatrix}$$

$$DC = T\begin{bmatrix} I & M \end{bmatrix}\begin{bmatrix} M \\ -I \end{bmatrix} = T(M - M) = 0$$

Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

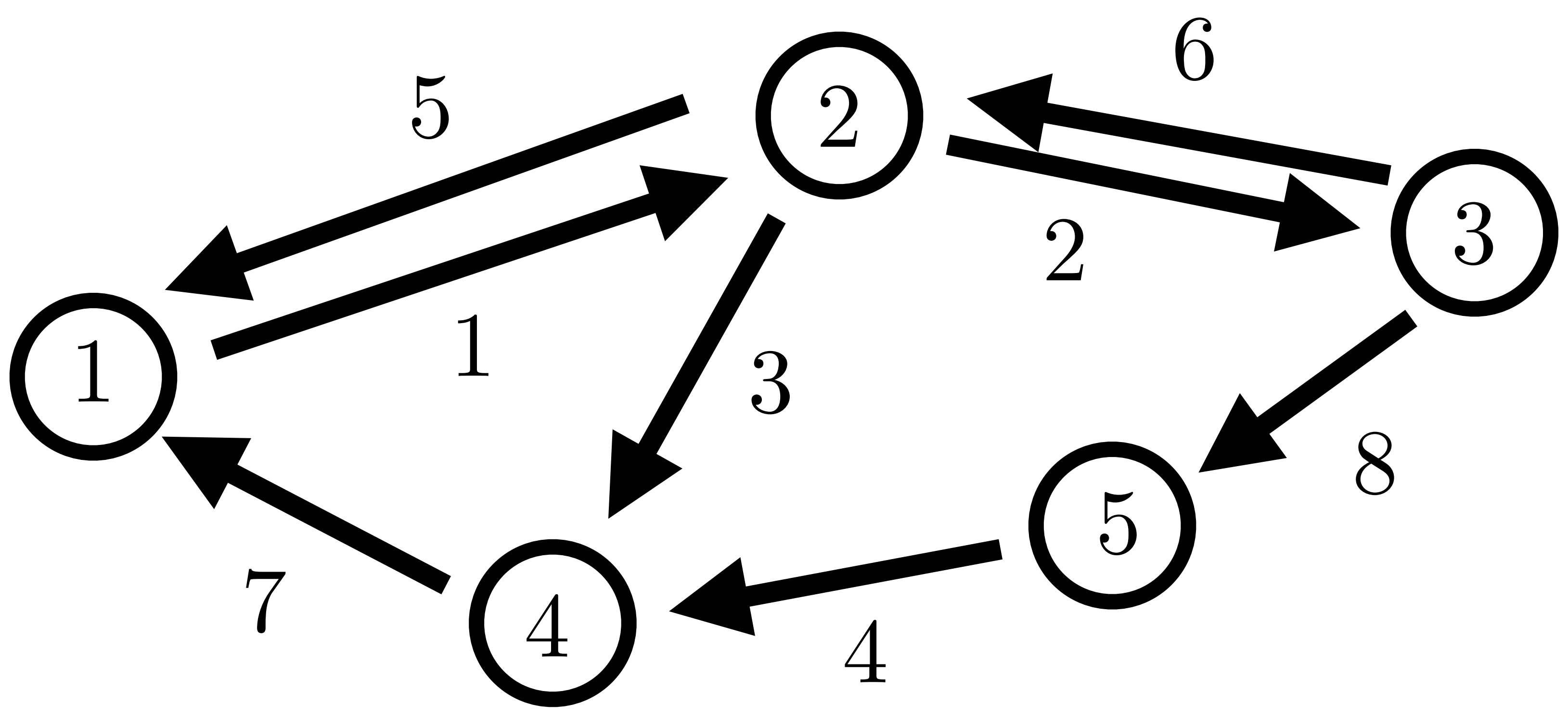
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges



Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis

$\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis

$\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$\left[\ C\ \right]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

Domain

Basis

$\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Nullspace D
$\mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

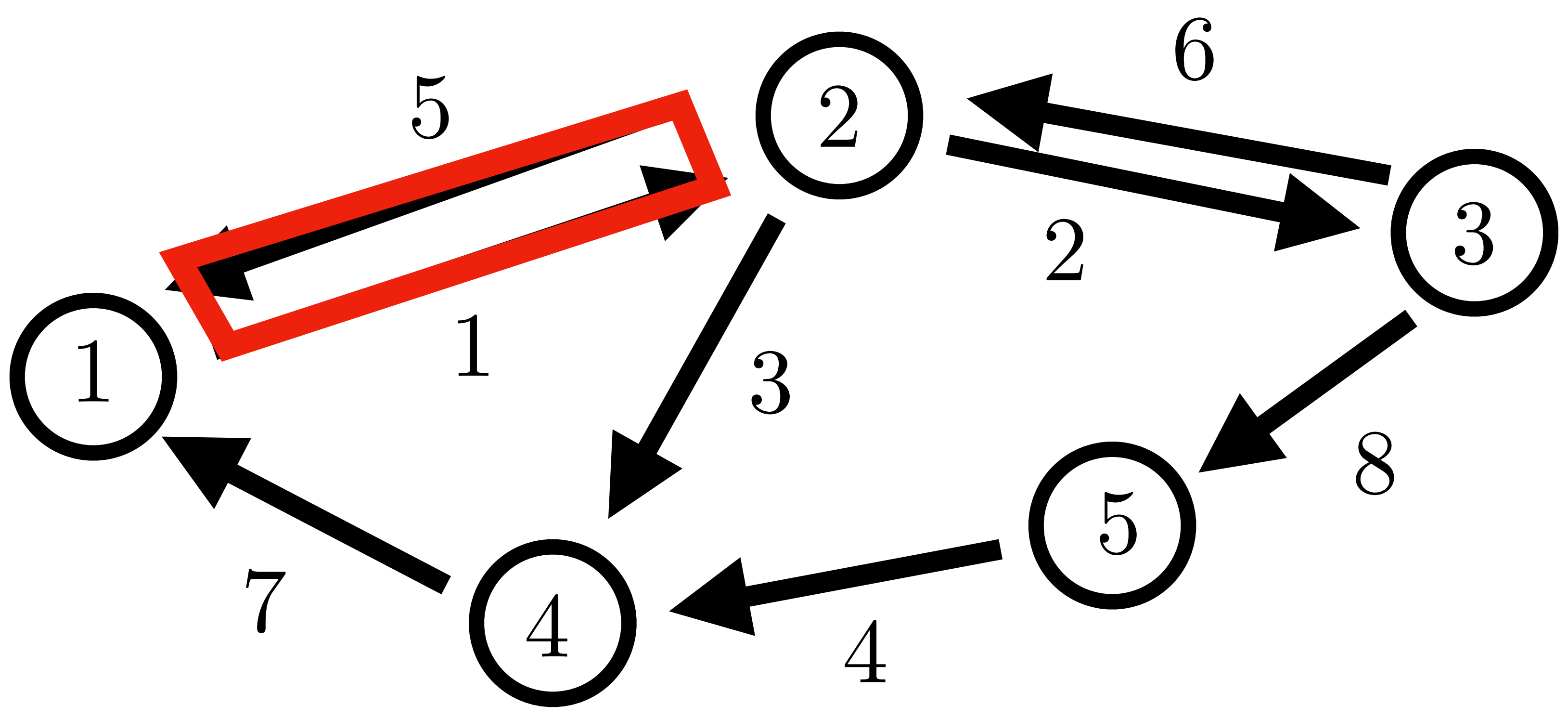
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges



Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D

dim = D

$\oplus^{\perp}$

Nullspace D^T

$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Range D^T

dim = D

$\oplus^{\perp}$

Nullspace D

$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

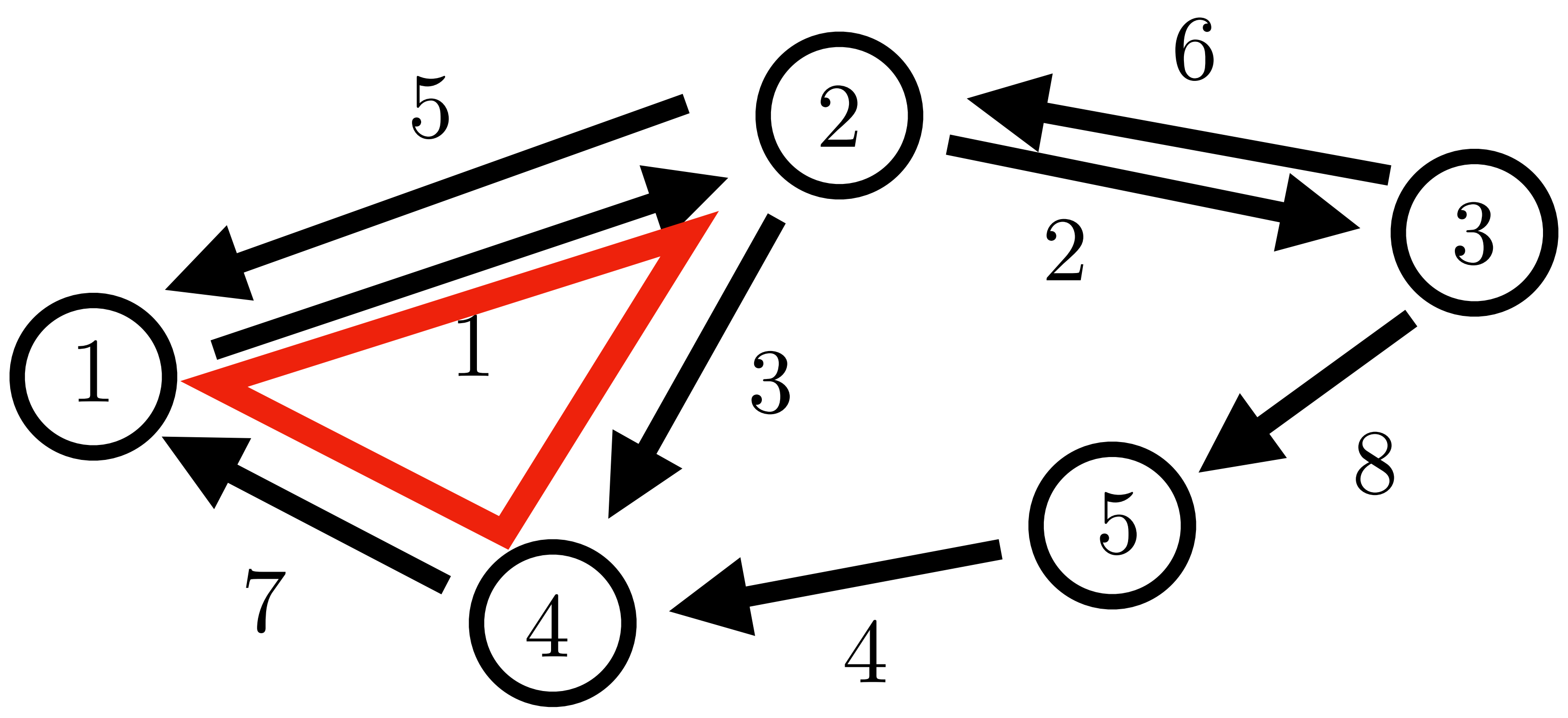
Basis **Cycles** $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

↕ vertices

← edges →

Right Nullspace

Co-Domain

Range D
dim = D

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

$\oplus^{\perp}$

Nullspace D^T
$\text{dim} = |\mathcal{V}| - \text{rk}D$

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

← Sign indicates if cycle goes with or against edge direction

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$

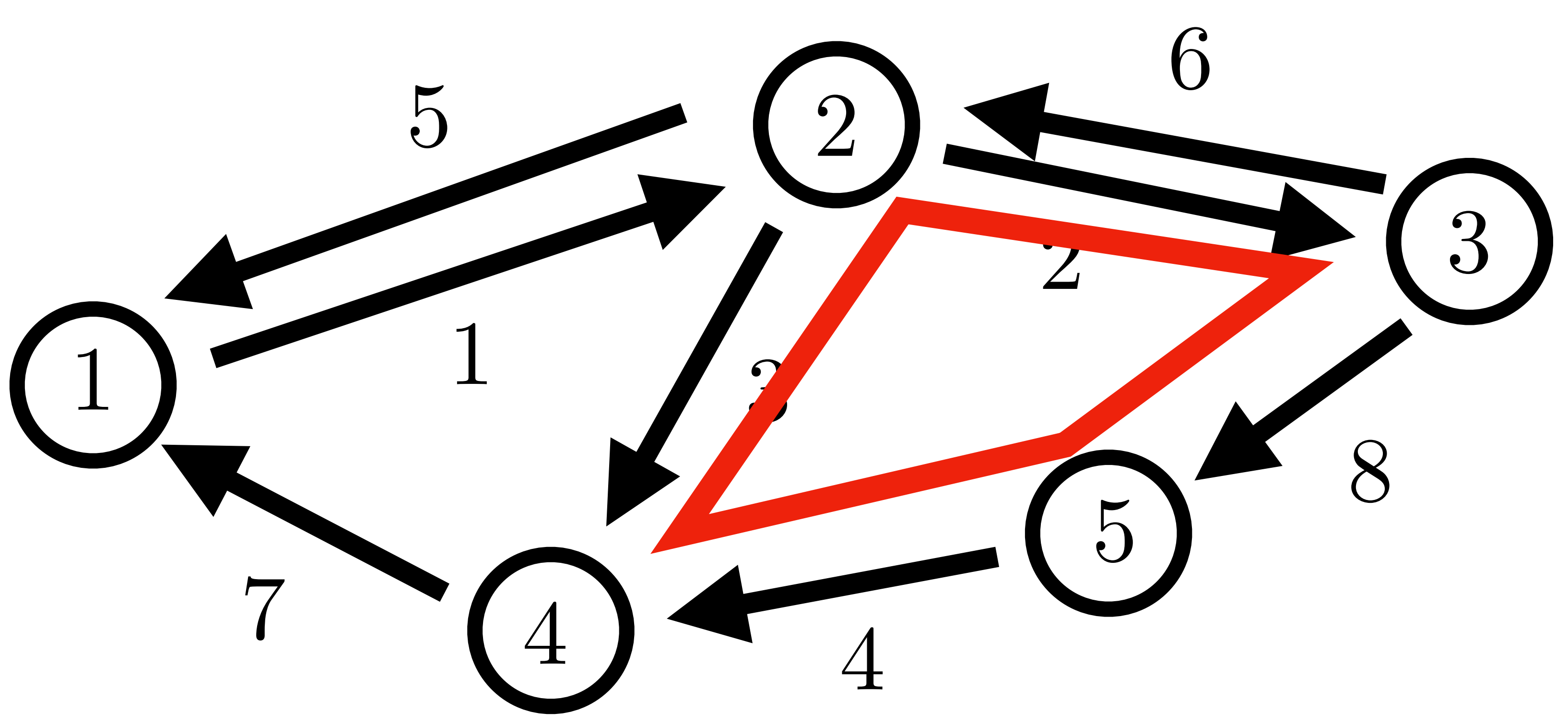
Nullspace D
$\text{dim} = |\mathcal{E}| - \text{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ **rank** D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

vertices

edges

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D $\quad$ **dim =** D

$\oplus^{\perp}$

Nullspace D^T $\quad \mathbf{dim} = |\mathcal{V}| - \mathbf{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ **Spanning Tree (Forest)**

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ **Constant vectors**

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$ $\quad$ **Range** D^T $\quad$ **dim =** D

$\oplus^{\perp}$

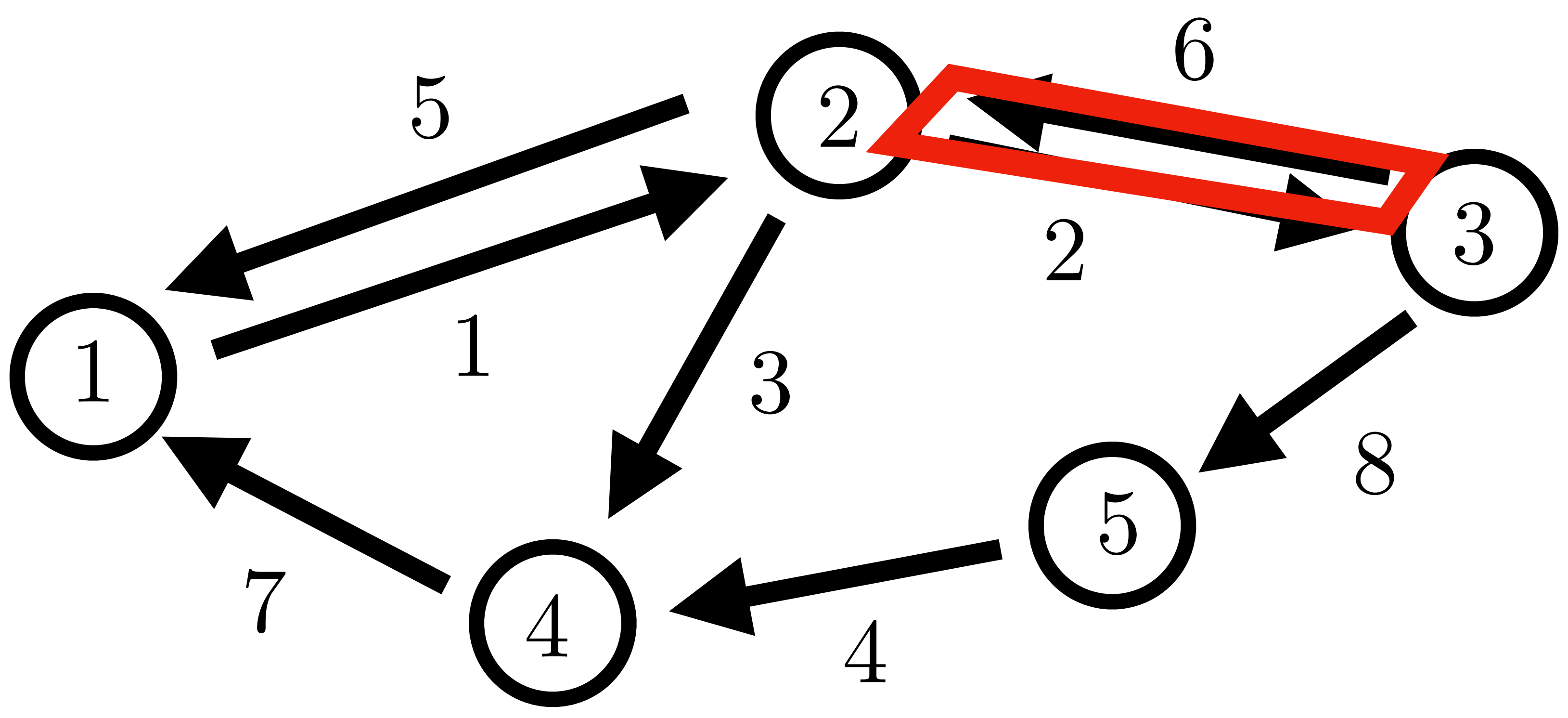
Basis **Cycles** $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$ $\quad$ **Nullspace** D $\quad \mathbf{dim} = |\mathcal{E}| - \mathbf{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

↑ vertices ↓

← edges →

Right Nullspace

$$Dx = 0$$

Conservation of flow at ea. node

$\Rightarrow$ x **is cycle flow** $\quad x = Cz$

Cycle indicator matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

← Sign indicates if cycle goes with or against edge direction

Co-Domain

Range D $\quad$ dim = D

$\oplus^{\perp}$

Nullspace D^T $\quad$ $\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbb{1} \\ | \end{bmatrix}$ Constant vectors

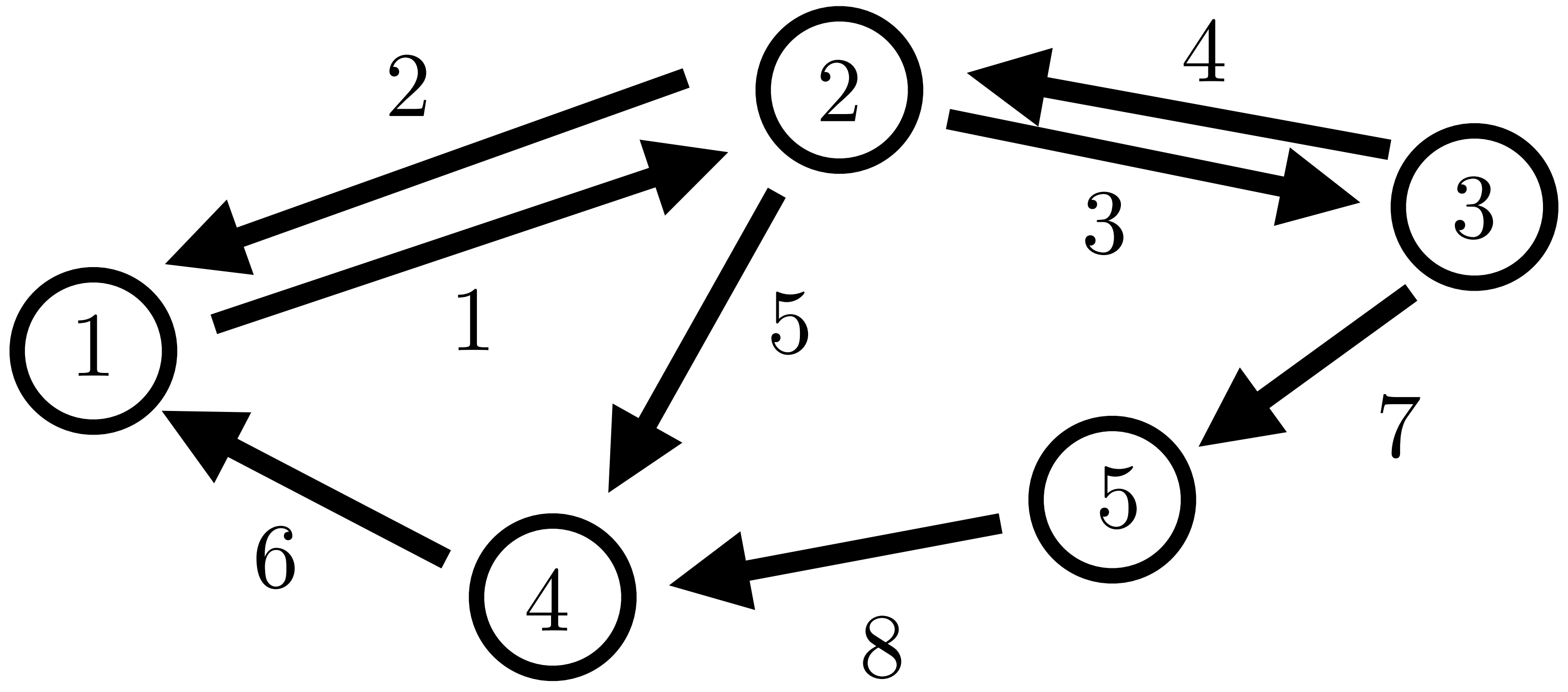
Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$ $\quad$ Range D^T $\quad$ dim = D

$\oplus^{\perp}$

Basis Cycles $\begin{bmatrix} | \\ C \\ | \end{bmatrix}$ $\quad$ Nullspace D $\quad$ $\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ rank D = $|\mathcal{V}| - n_c$

$$\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1\\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1\end{bmatrix} = \mathbf{0}$$

Left Nullspace

$$\mathbf{1}^T D = 0$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\dim = |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

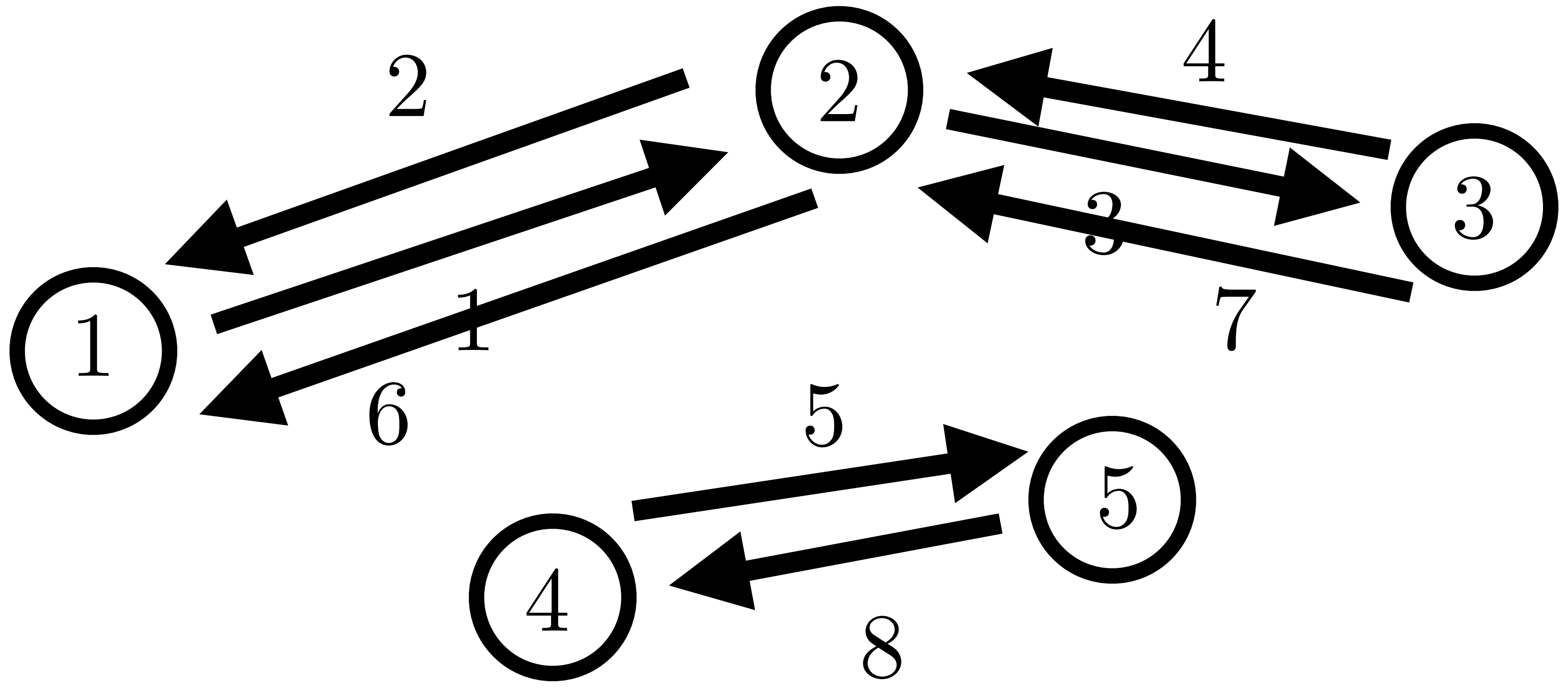
Nullspace D
$\dim = |\mathcal{E}| - \mathrm{rk} D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad$ rank D = $|\mathcal{V}| - n_c$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \left[\begin{array}{cccc|c|cc|c} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] = \mathbf{0}$$

Left Nullspace

$$\mathbf{1}^T D = 0$$

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
$\mathrm{dim} = |\mathcal{V}| - \mathrm{rk}D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \mathbf{1} \\ | \end{bmatrix}$ Constant vectors

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$ Range D^T
dim = D

$\oplus^{\perp}$

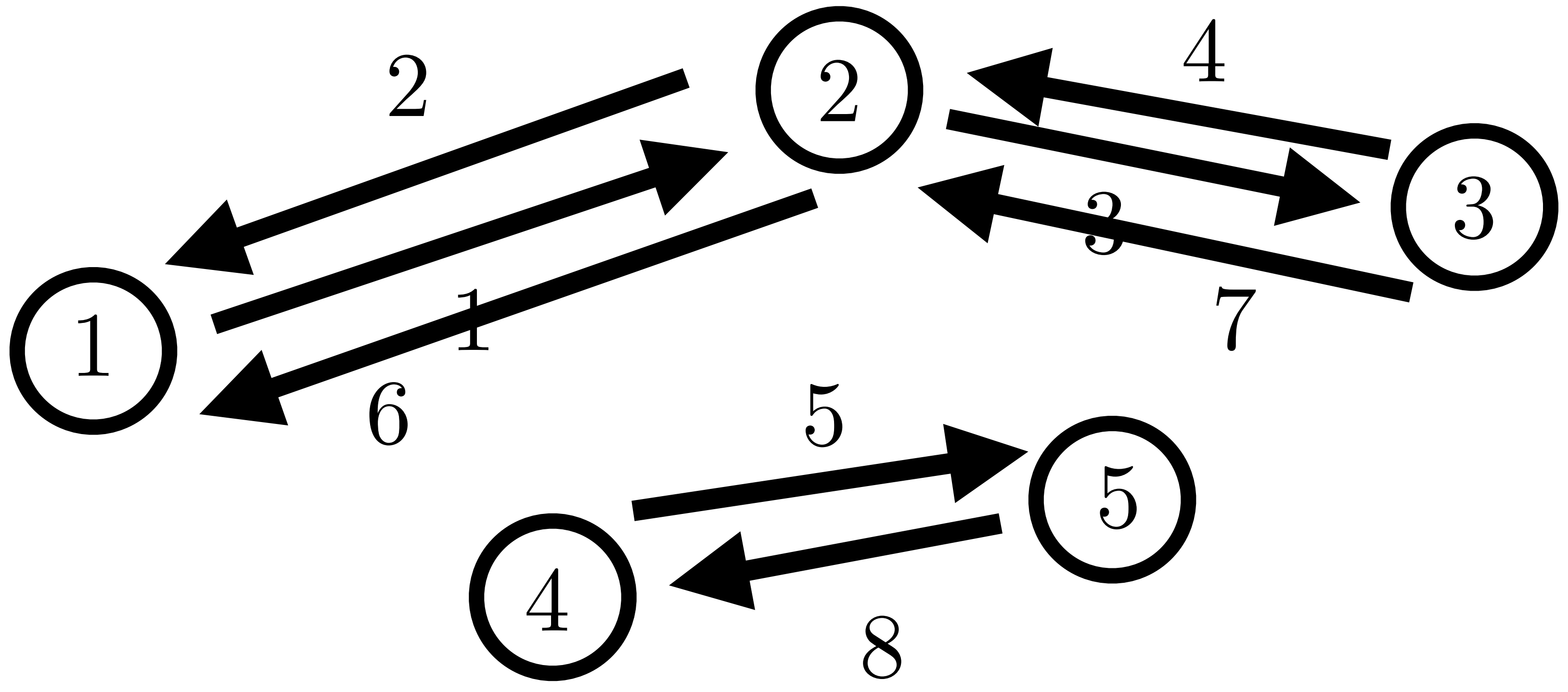
Basis
Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$ Nullspace D
$\mathrm{dim} = |\mathcal{E}| - \mathrm{rk}D$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ rank D = $|\mathcal{V}| - n_c$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

Left Nullspace

Co-Domain

Range D
dim = D

$\oplus^{\perp}$

Nullspace D^T
dim $= |\mathcal{V}| - \mathrm{rk} D$

Basis $\begin{bmatrix} | \\ T \\ | \end{bmatrix}$ Spanning Tree (Forest)

Basis $\begin{bmatrix} | \\ \bar{\mathbf{1}} \\ | \end{bmatrix}$ Constant vectors

$$\underbrace{\begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix}}_{\bar{\mathbf{1}}^T} \begin{bmatrix} D \end{bmatrix} = \mathbf{0}$$

dim = num connected components

Domain

Basis $\begin{bmatrix} I \\ M^T \end{bmatrix}$

Range D^T
dim = D

$\oplus^{\perp}$

Basis

Cycles $\begin{bmatrix} M \\ -I \end{bmatrix}$

Nullspace D
dim $= |\mathcal{E}| - \mathrm{rk} D$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

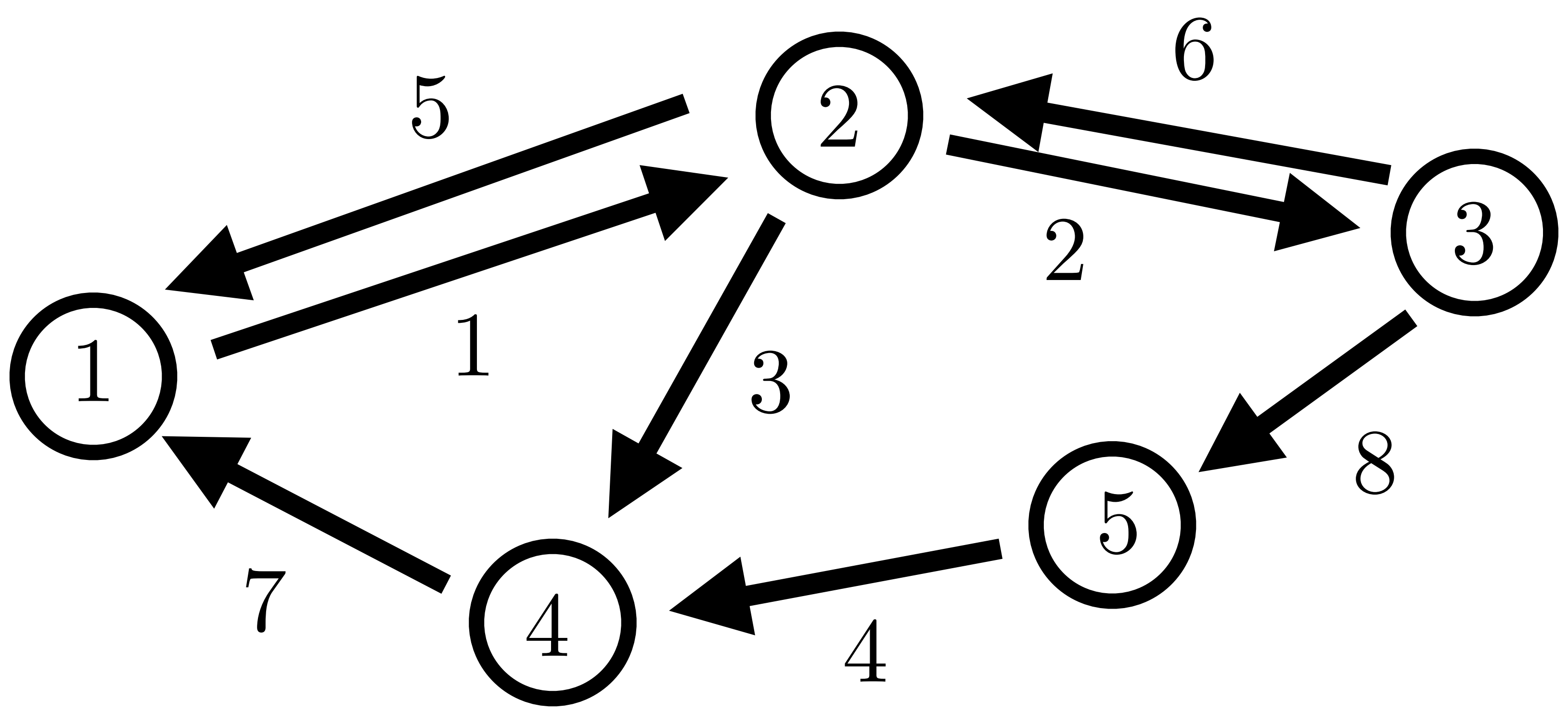
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**



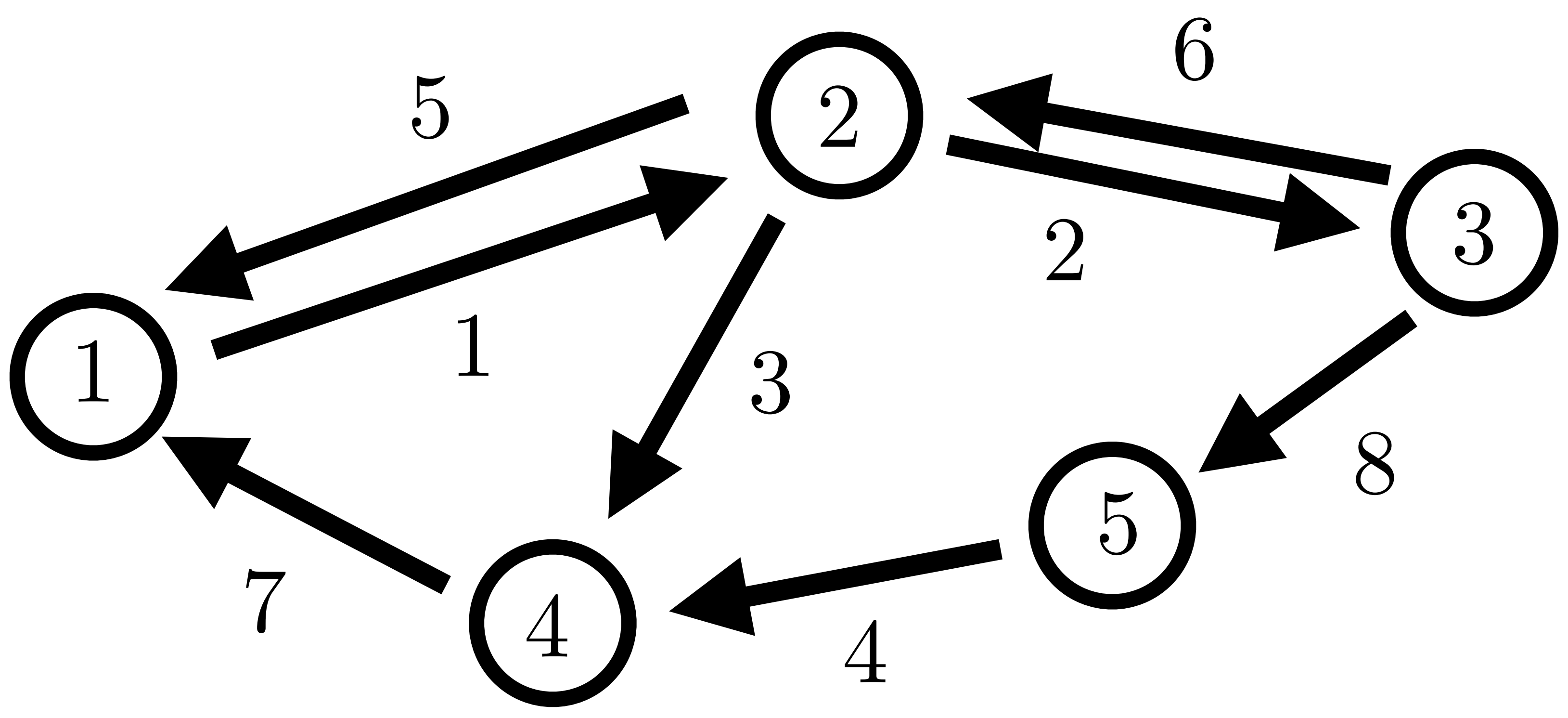
Review: Shape Matrices

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

cols…

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

Inner products of columns

“Relative geometry of columns”

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

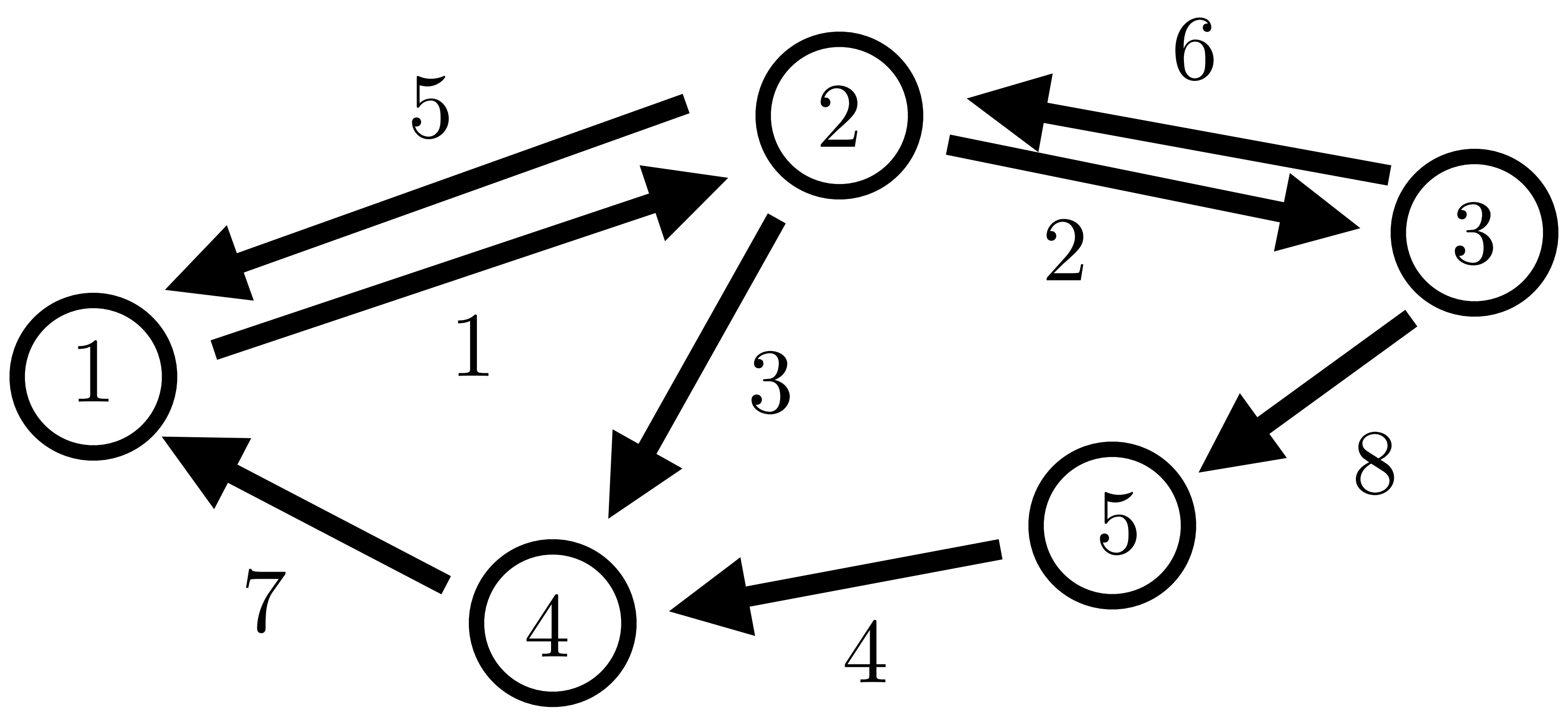
rows…

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

Inner products of rows

“Relative geometry of rows”

Graph Laplacians



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

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$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

cols...

$$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

rows...

$$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

RA rotate columns of A…

….relative geometry stays the same.

$$(RA)^T(RA) = A^T R^T R A = A^T A$$

AR rotate rows of A…

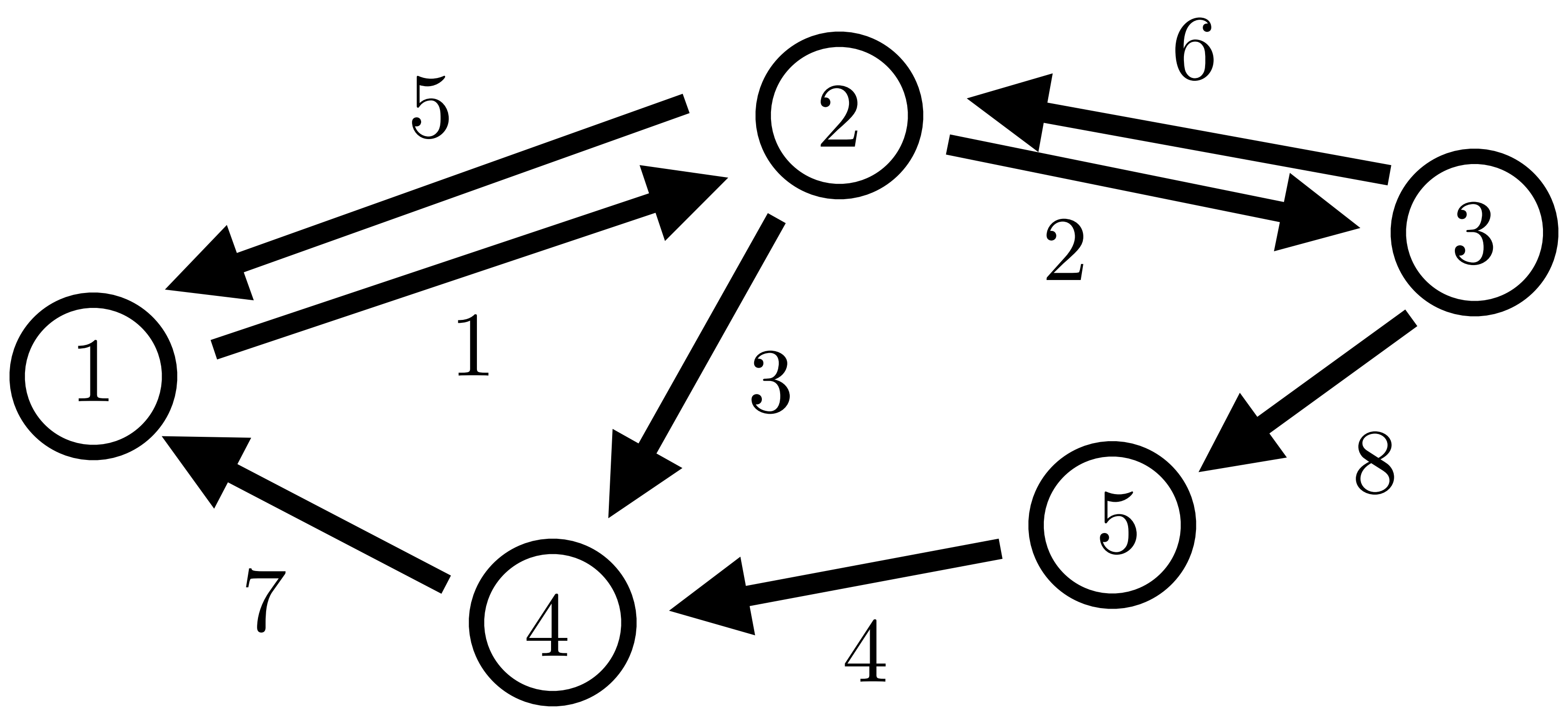
….relative geometry stays the same.

$$(AR)(AR)^T = ARR^T A^T = AA^T$$

Graph Laplacians

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$A^T A$ "Shape" of the columns of A

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Graph Laplacians

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

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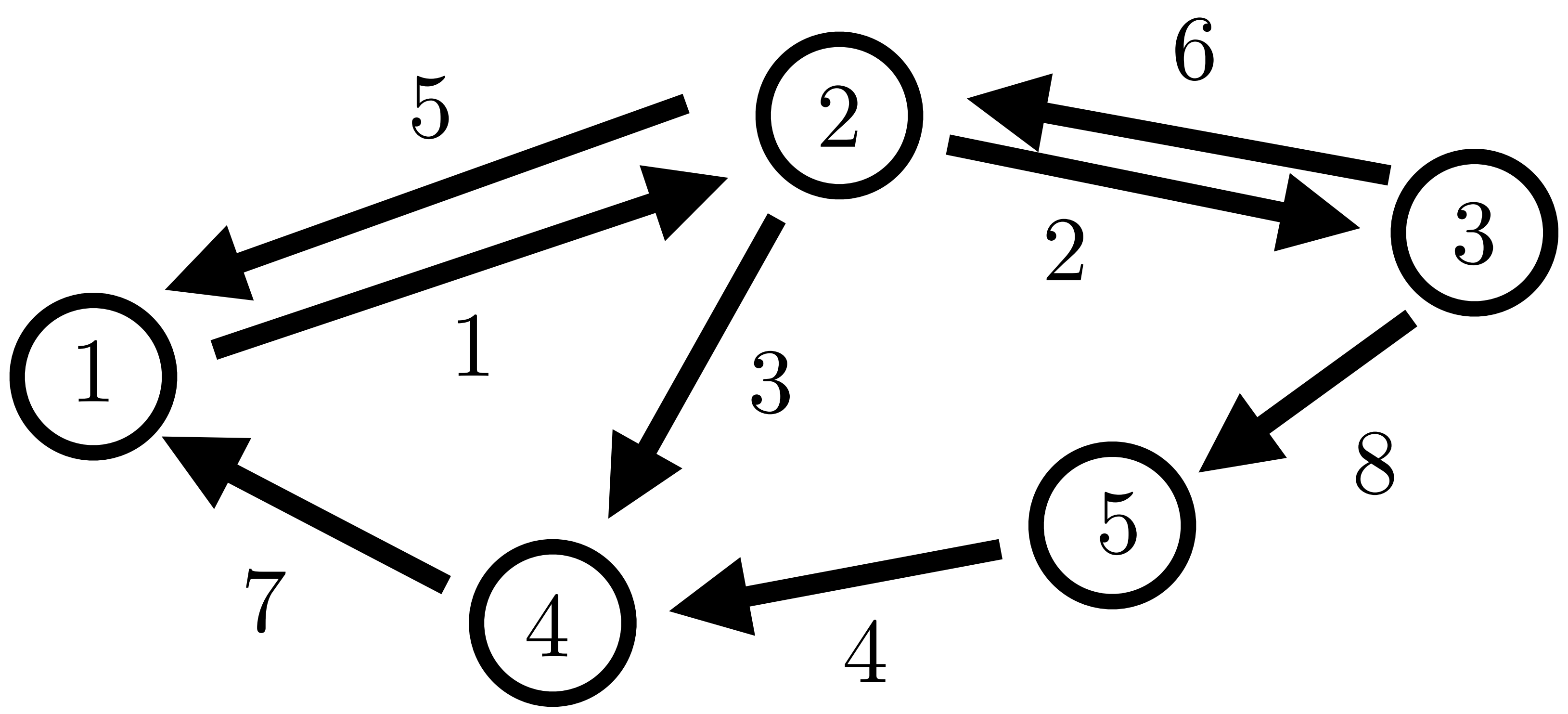

Review: Shape Matrices

$A^T A$ ~~“Shape” of the columns of A~~

Graph Laplacians

Graph:

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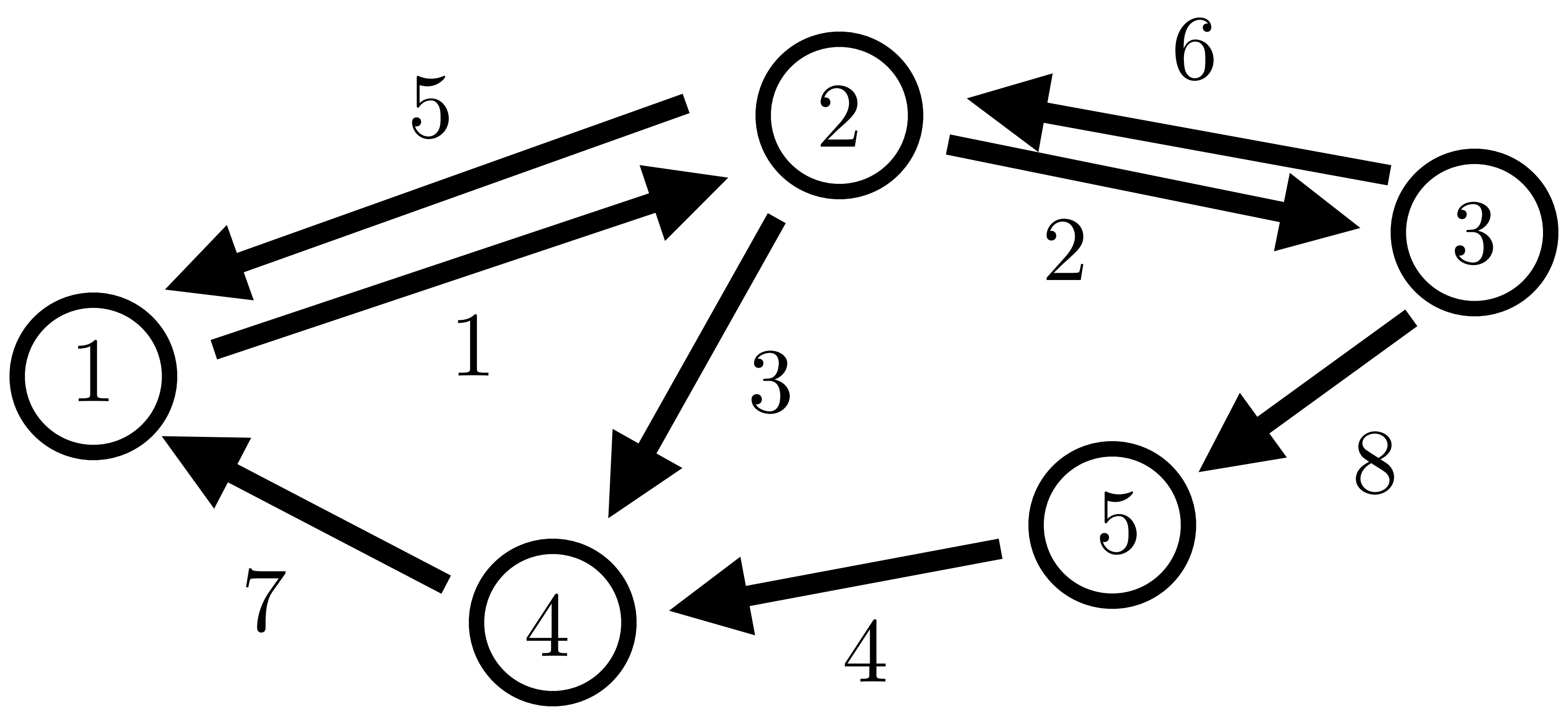


Review: Shape Matrices

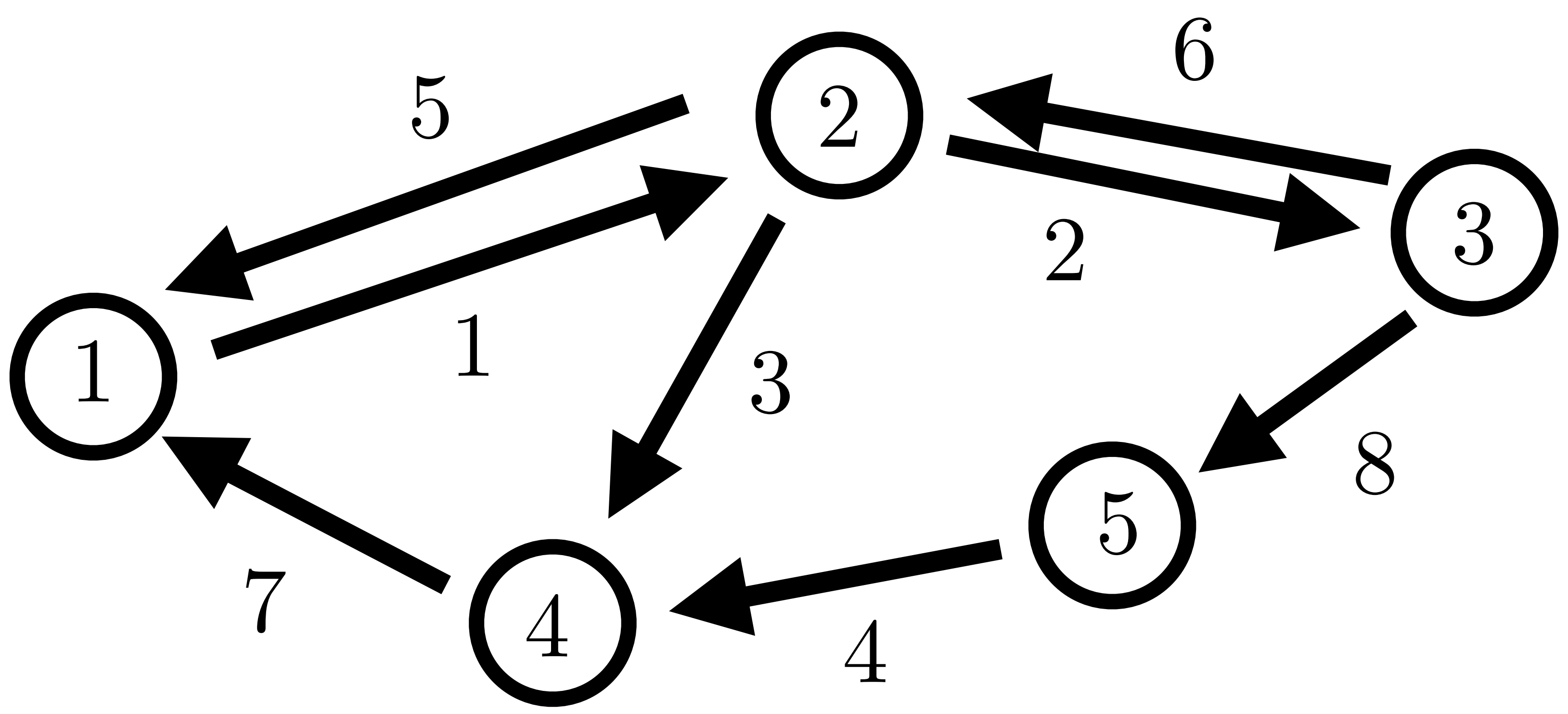
$(A^T A)^{1/2}$ “Shape” of the columns of A

More Accurate

Graph Laplacians

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$



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Review: Shape Matrices

$(A^T A)^{1/2}$ “Shape” of columns $(AA^T)^{1/2}$ “Shape” of rows

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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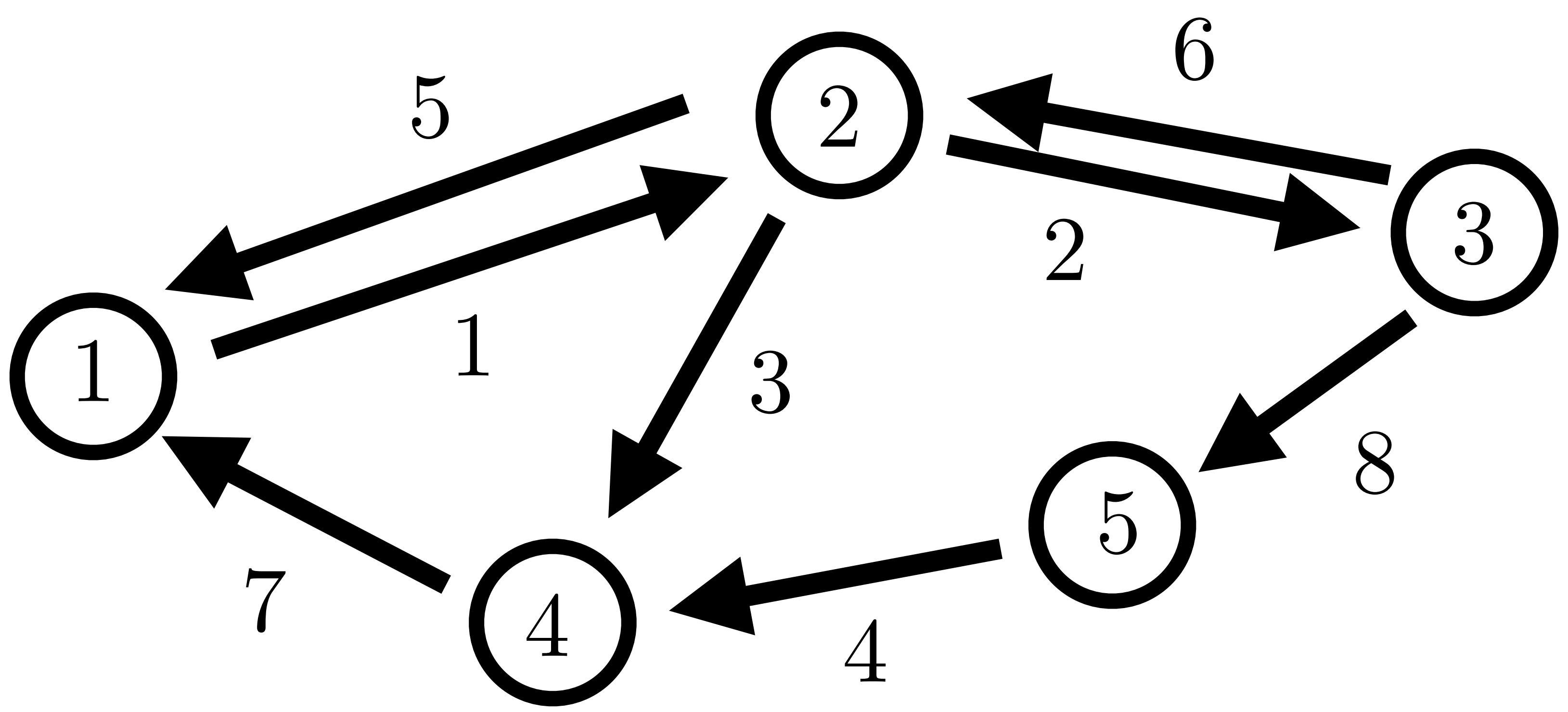

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Review: Shape Matrices

$(A^T A)^{1/2}$ “Shape” of columns $\quad (AA^T)^{1/2}$ “Shape” of rows

Analogy: $z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z| e^{i\phi}$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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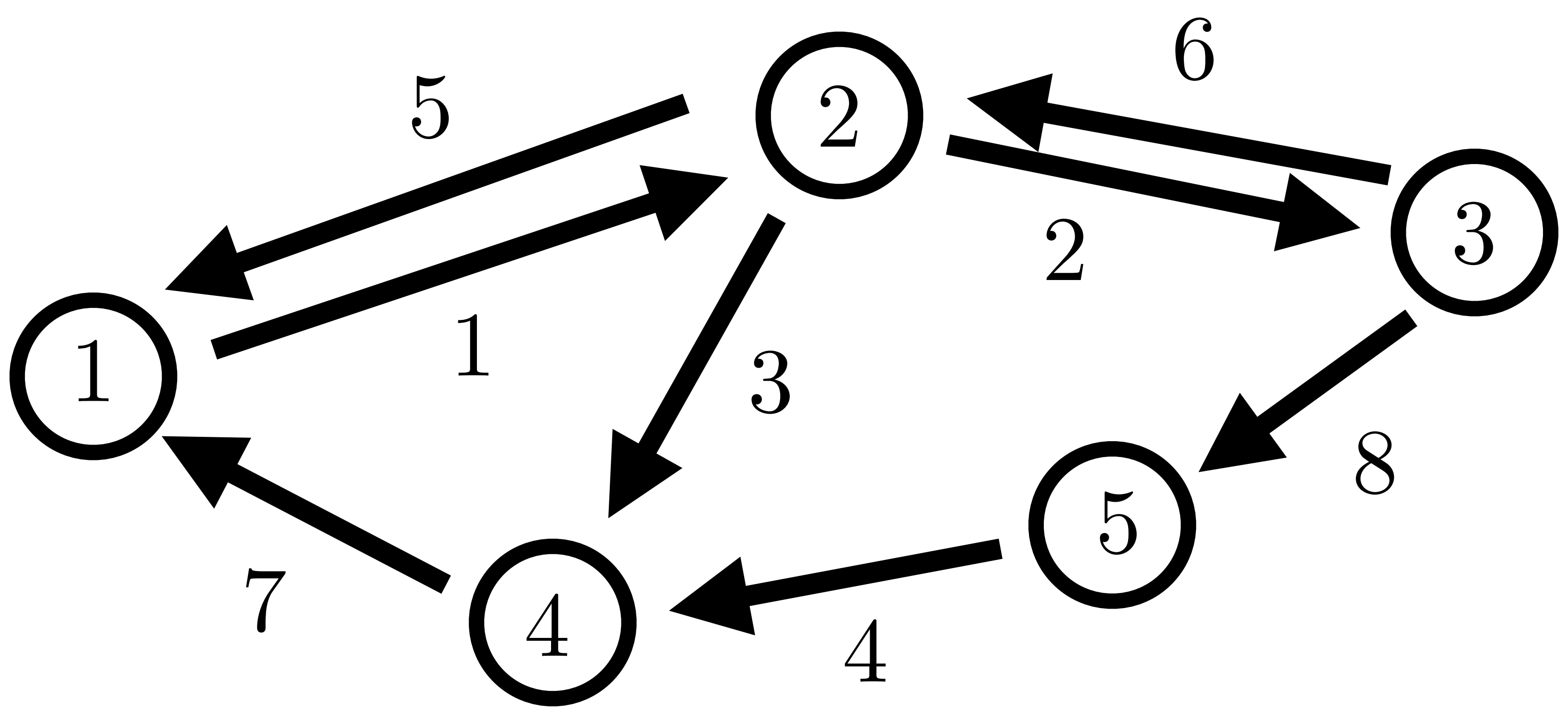

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $(AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C}$ $|z| = \sqrt{z^* z}$ $z = |z| e^{i\phi}$

Polar Decomposition

$$A = \underbrace{A(A^T A)^{-1/2}}_{\text{Rotation}} \cdot \underbrace{(A^T A)^{1/2}}_{\text{PSD "shape"}}$$

"Column version"

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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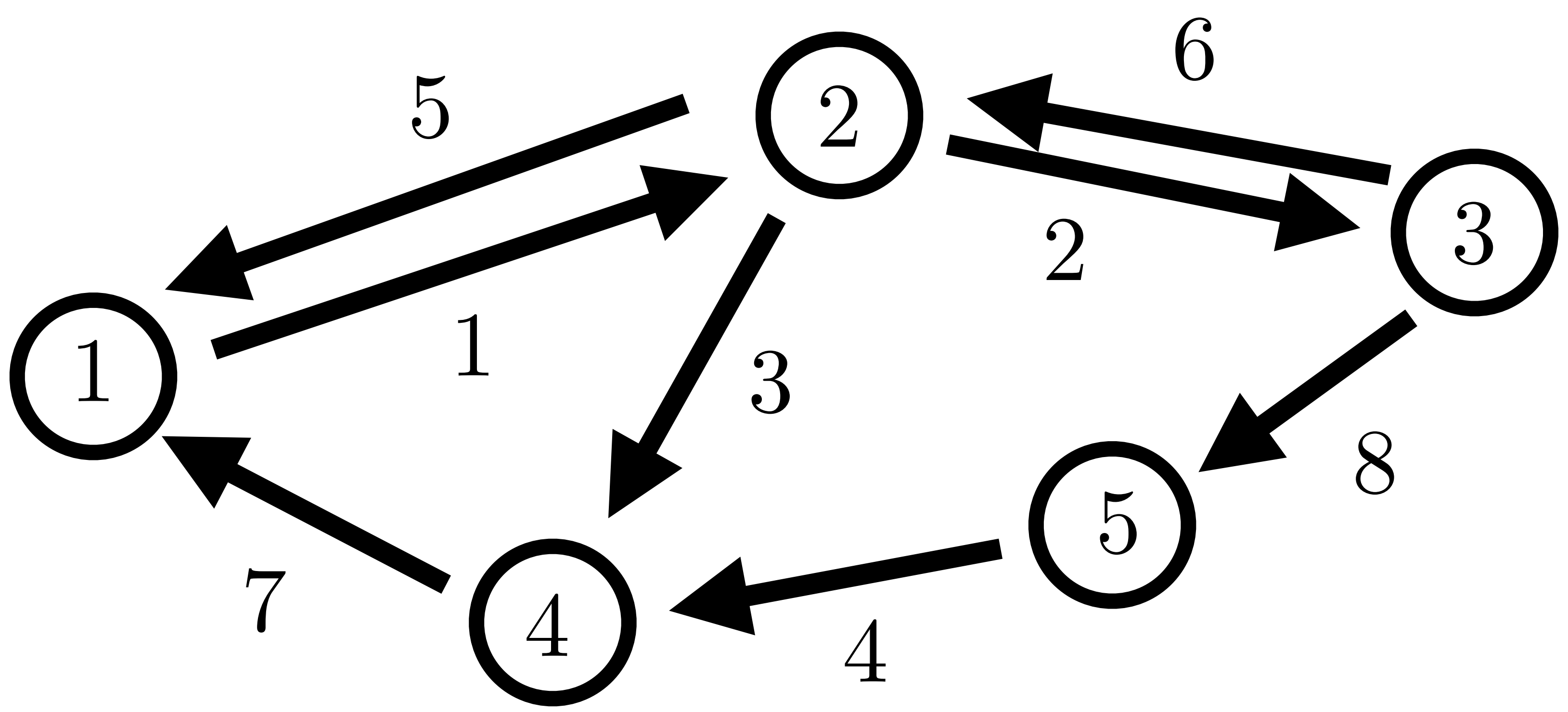

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C}$ $\quad |z| = \sqrt{z^* z}$ $\quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation $\quad$ PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" $\quad$ Rotation

"Row version"

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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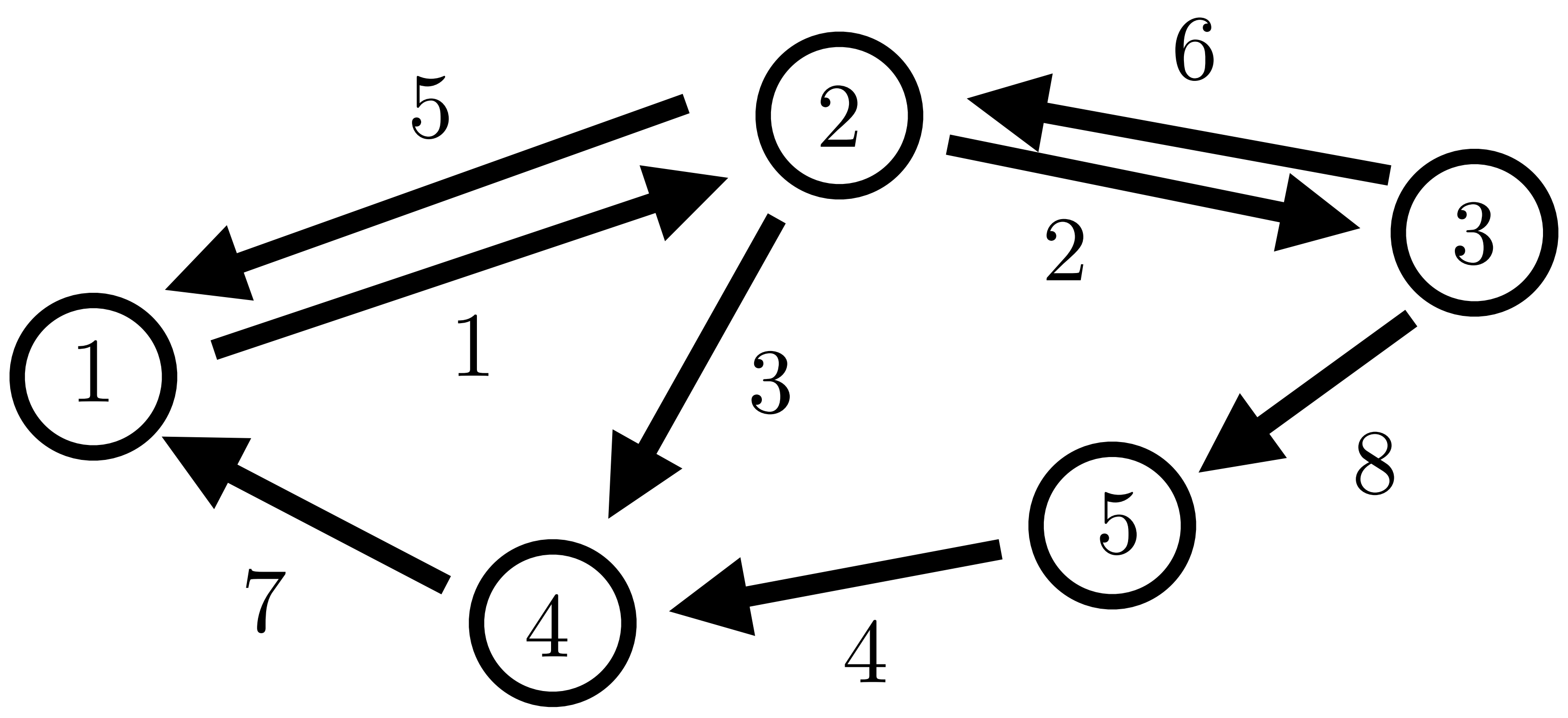


Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C}$ $\quad |z| = \sqrt{z^* z}$ $\quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation $\quad$ PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" $\quad$ Rotation

"Row version"

Checking rotation…

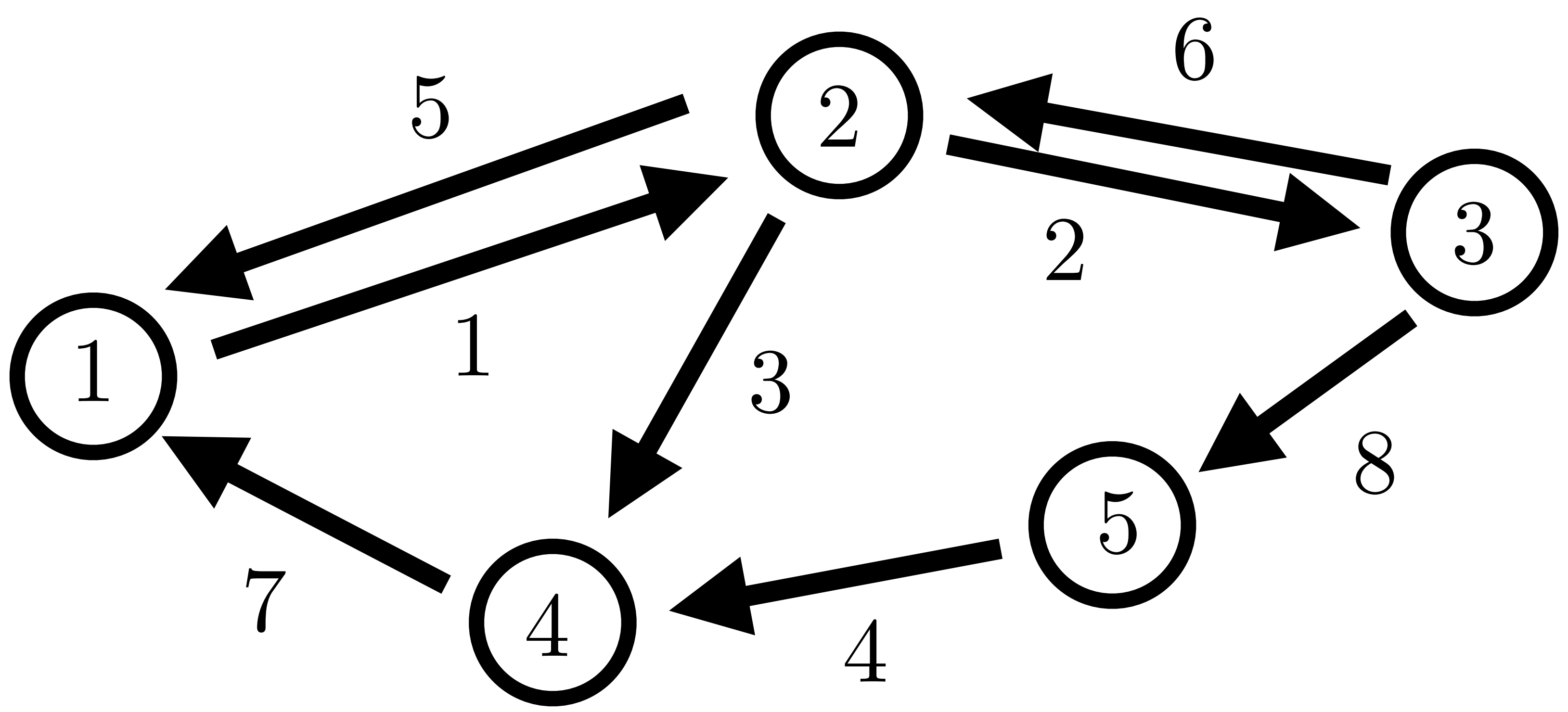
$$(A^T A)^{-1/2} A^T A (A^T A)^{-1/2} = I$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



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General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Sym/PSD Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes $\quad (A^TA)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \quad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$

Nullspace $\quad$ Null space A = Null space A^TA $\quad$ Null space A^T = Null space AA^T

Rank $\quad \text{rank}(A) = \text{rank}(A^T) = \text{rank}(A^TA) = \text{rank}(AA^T)$

Symmetric matrix $\quad S \in \mathbb{R}^{n \times n}$ has orthonormal eigenvectors

Positive semi-definite $\quad x^TSx \geq 0 \quad \forall x \iff \lambda_i \geq 0 \quad \lambda_i \in \text{eig}(S)$

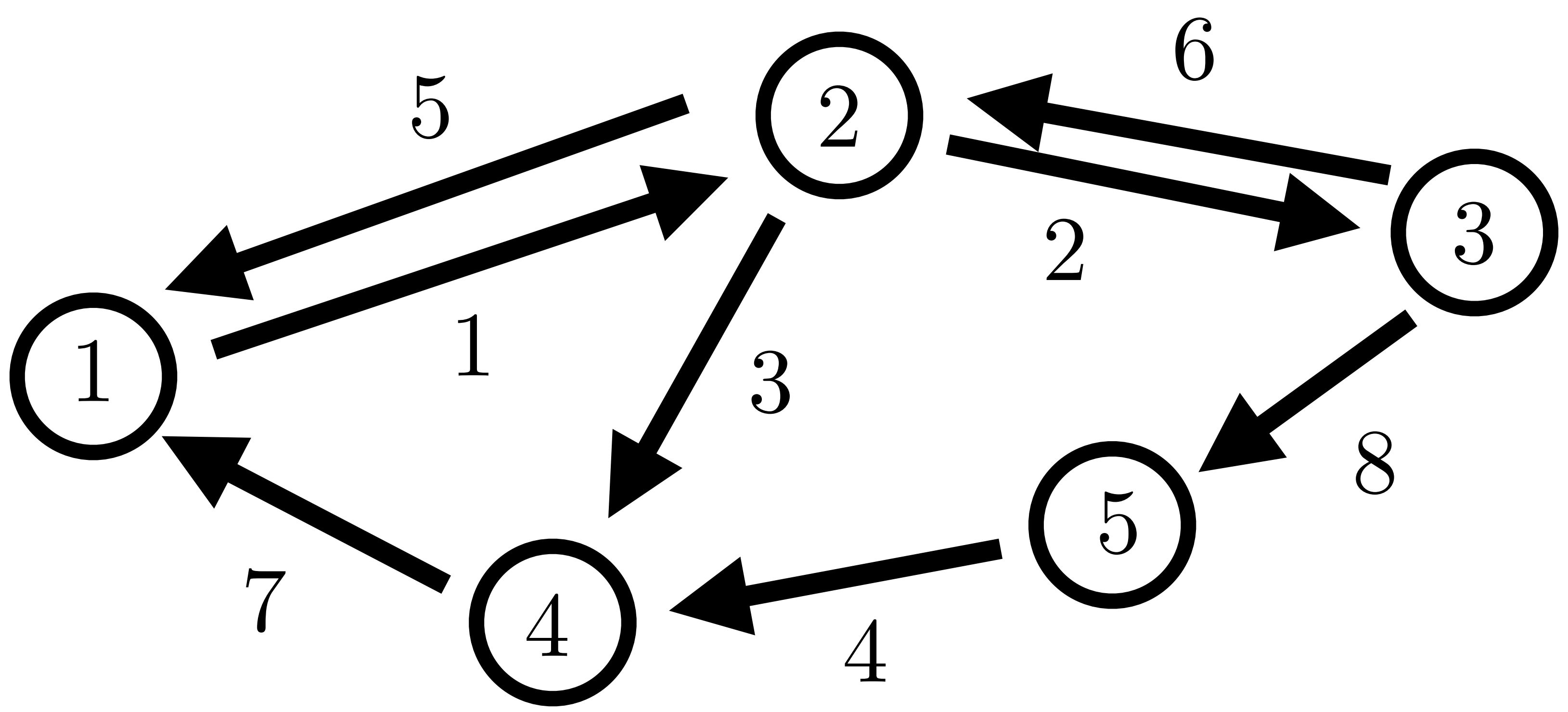
$S \succeq 0 \quad A^TA,\ AA^T,\ (A^TA)^{1/2},\ (AA^T)^{1/2}$ all PSD

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation $\quad$ PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" $\quad$ Rotation

"Row version"

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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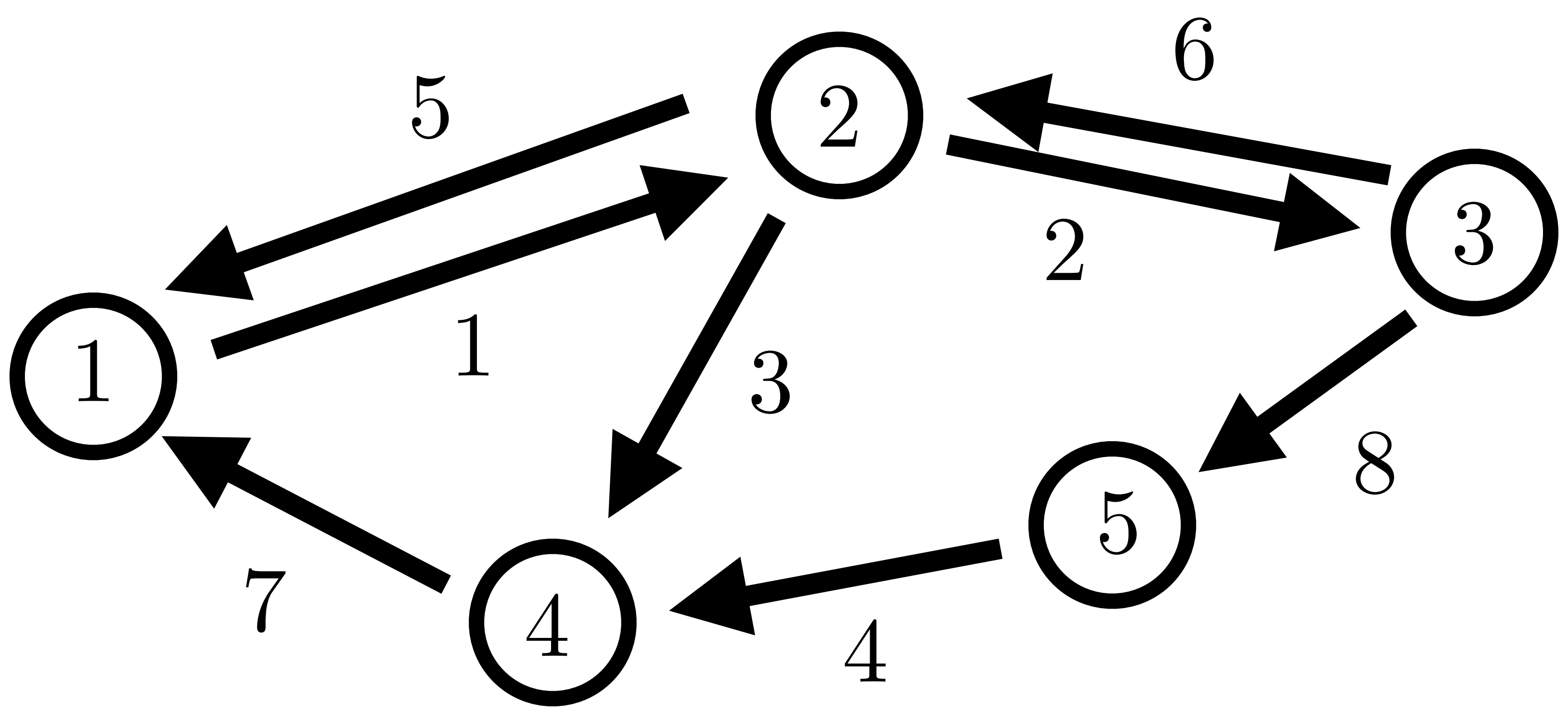

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General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

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Review: Shape Matrices

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EVD of Shapes

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Polar Decomposition

$$A = UV^T \cdot V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

"Column version"

Rotation $\qquad$ PSD "shape"

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T \cdot UV^T$$

"Row version"

PSD "shape" $\qquad$ Rotation

Graph Laplacians

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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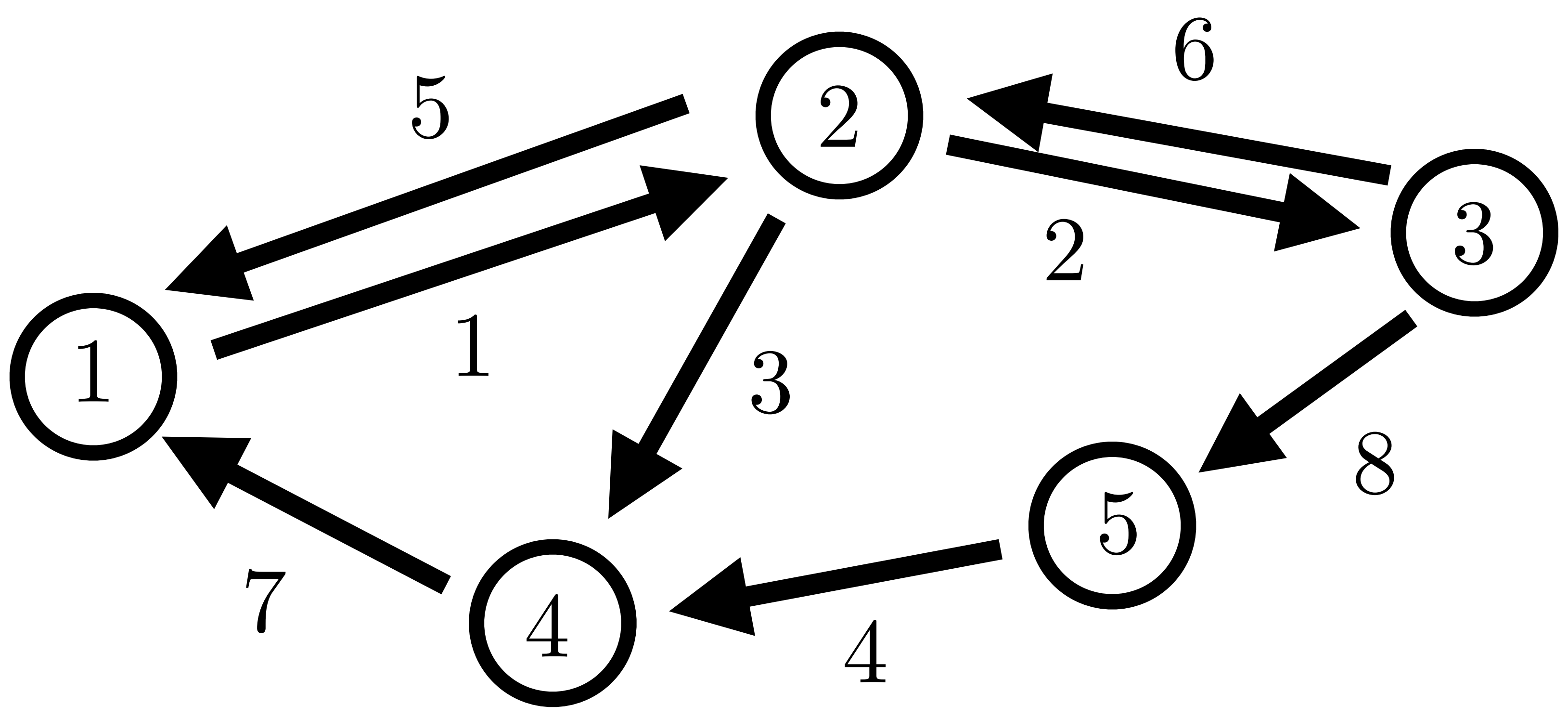

Review: Shape Matrices

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EVD of Shapes

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Singular Value Decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians

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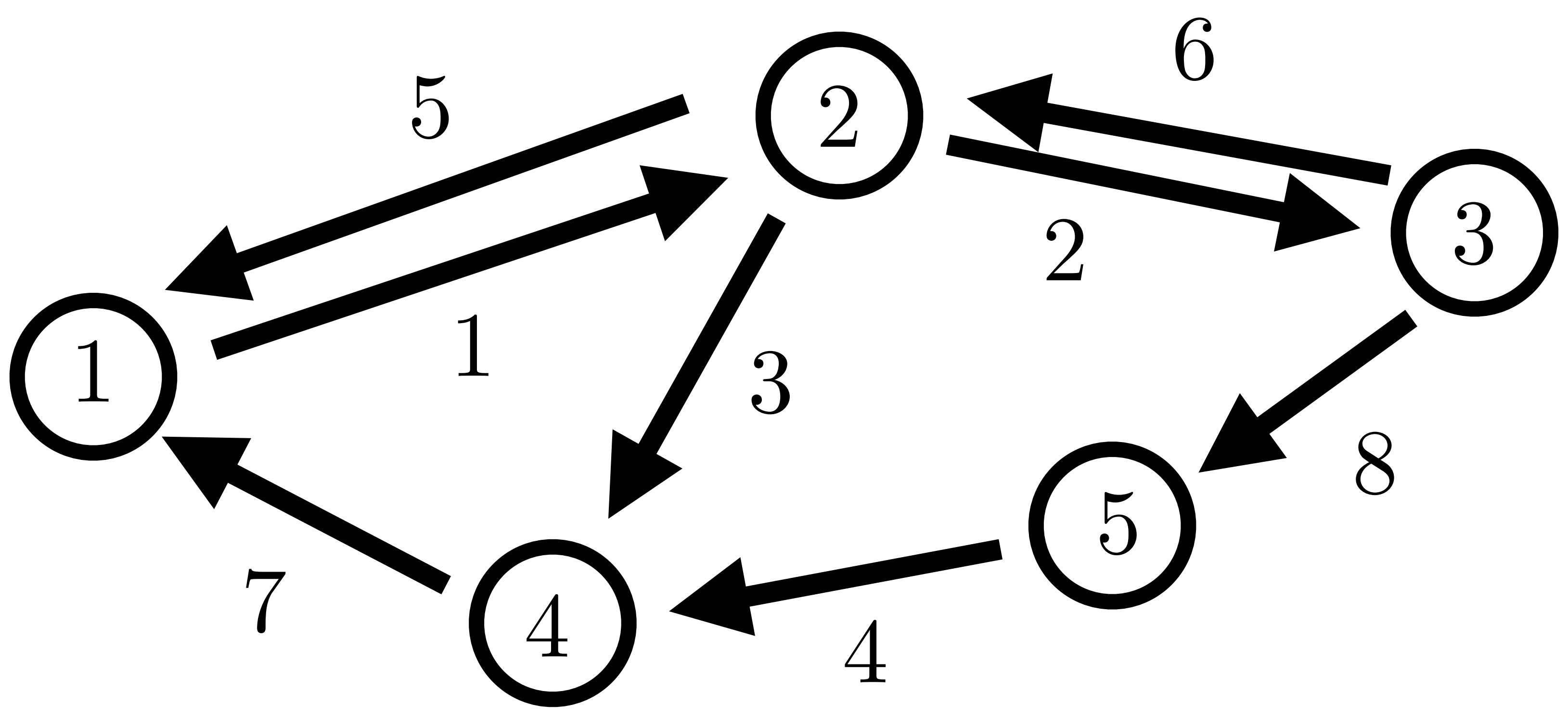

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Singular Value Decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$= \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix}$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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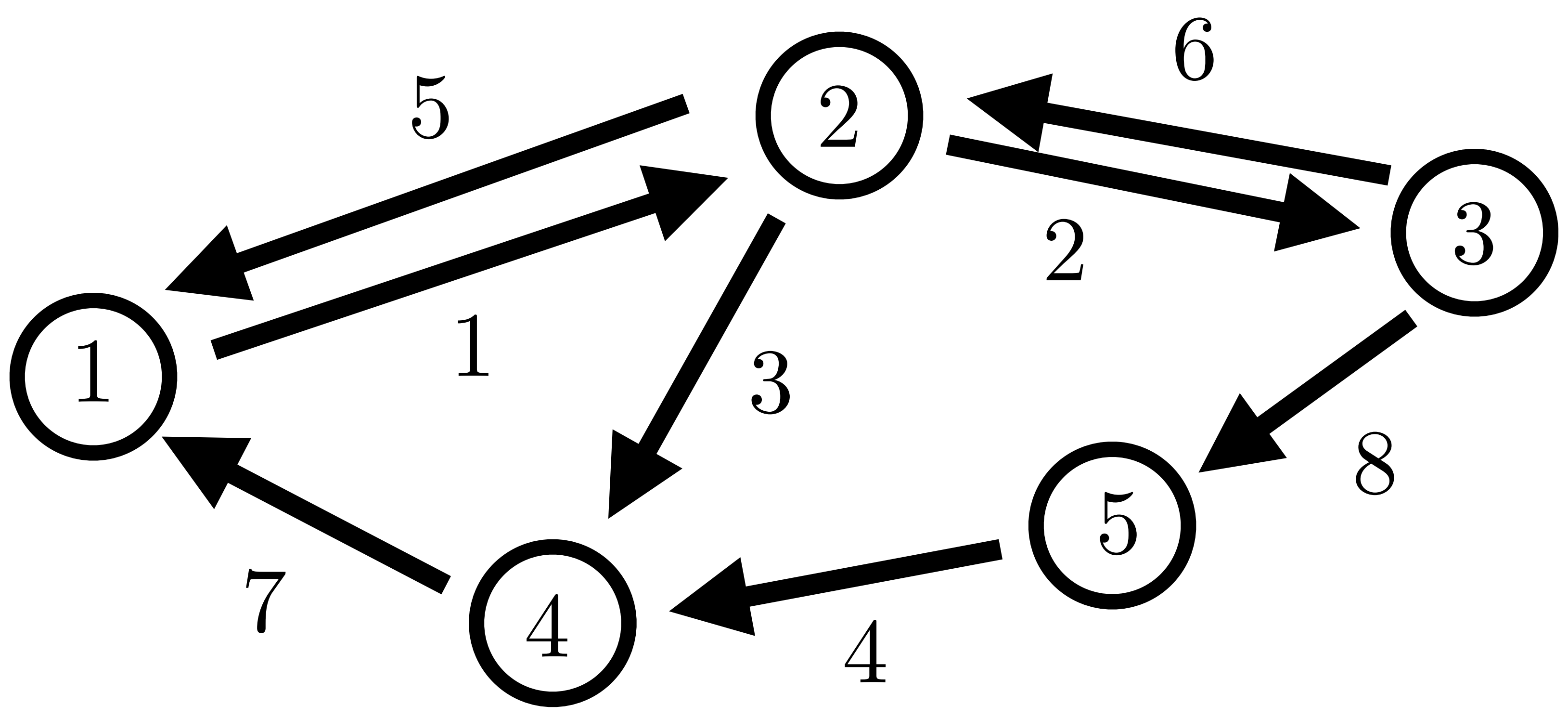
Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Singular Value Decomposition

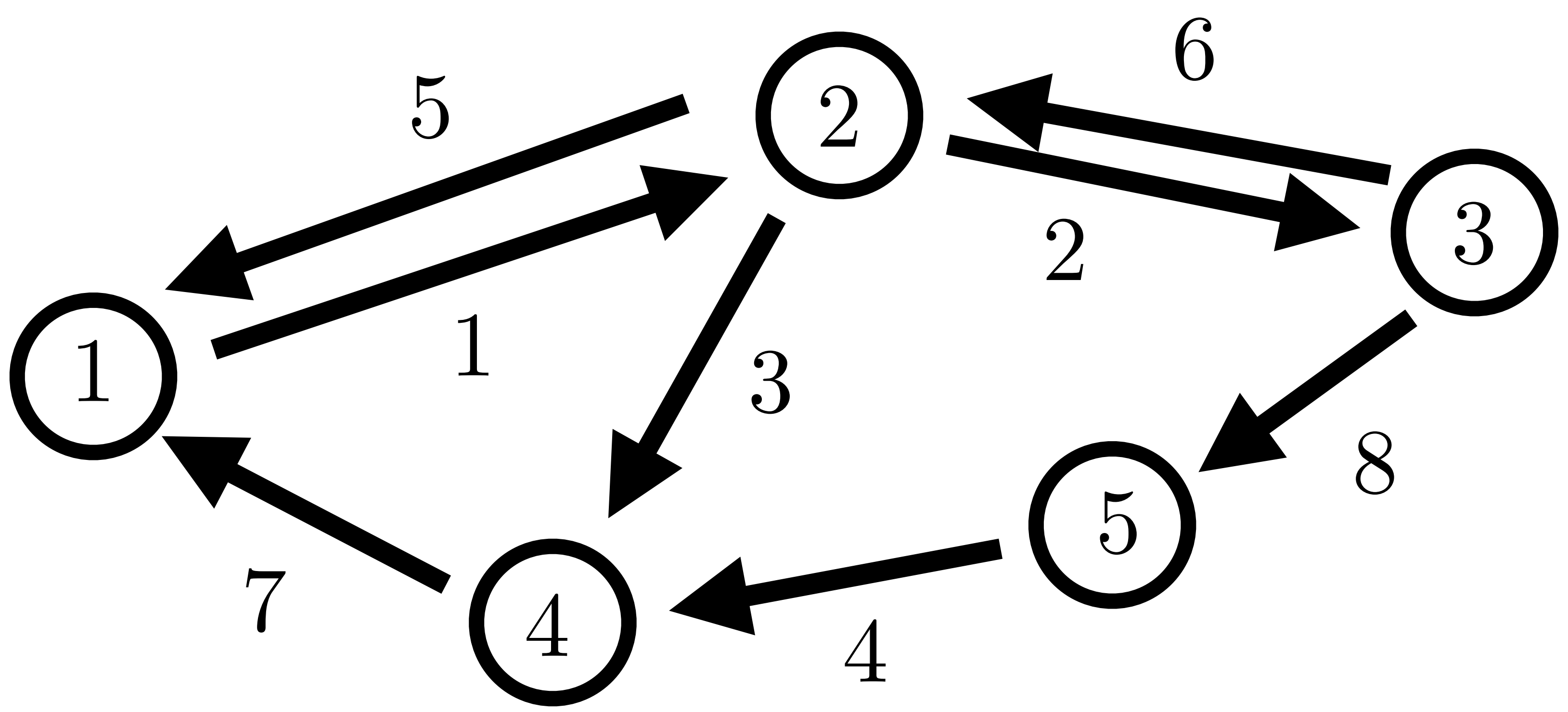
$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$= \begin{bmatrix} | & | \\ U' & U'' \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & V'^T & - \\ - & V''^T & - \end{bmatrix}$$

$$U' = AV'\Sigma^{-1} \qquad V'^T = \Sigma^{-1} U'^T A$$

for singular vectors w/ non-zero values

Graph Laplacians



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Laplacian $L = DD^T$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

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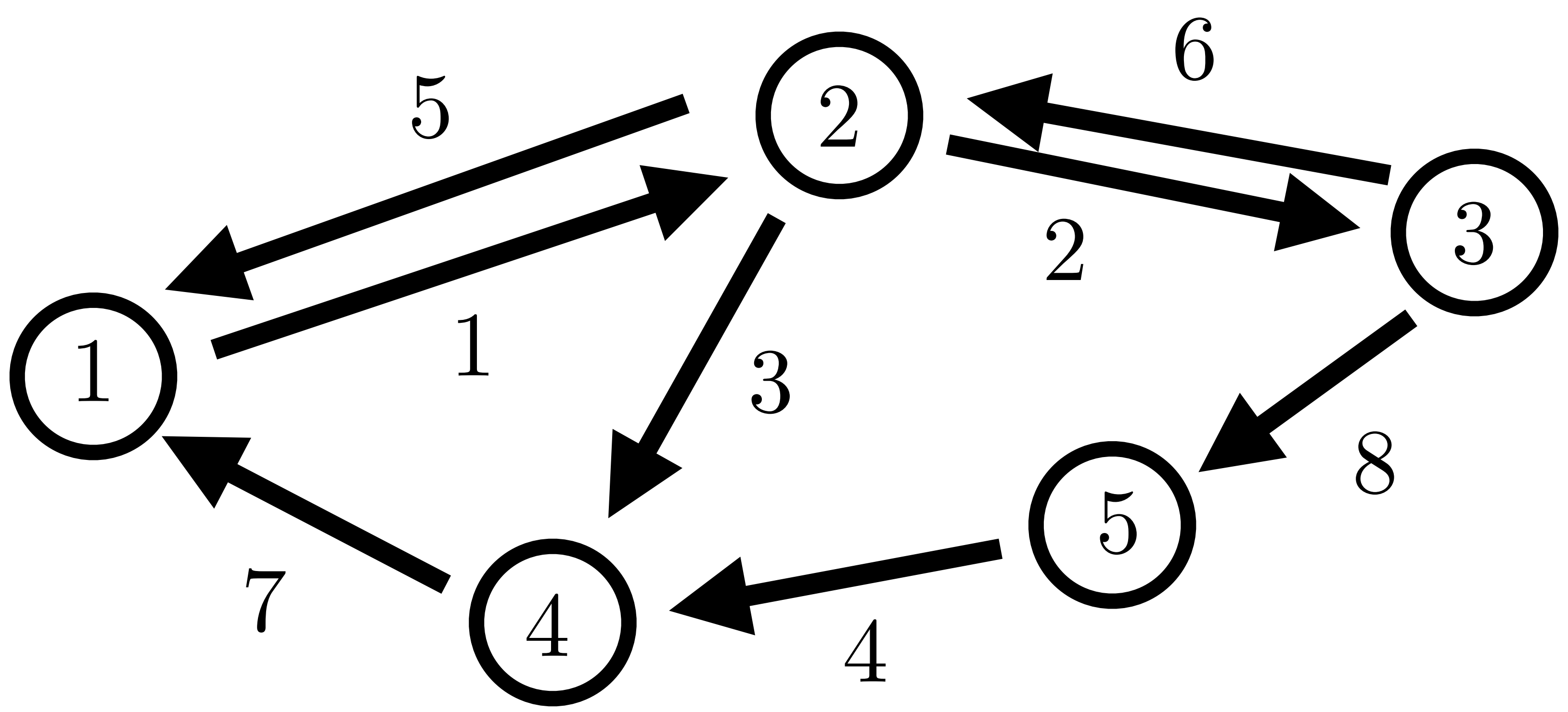

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Laplacian $L = DD^T$

Action:

$$Lu = \underbrace{\Big[\ D\ \Big] \underbrace{\Big[\ D^T\ \Big] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges}}}_{\text{... summed resulting tension on nodes}}$$

"heights" of nodes

Graph Laplacians

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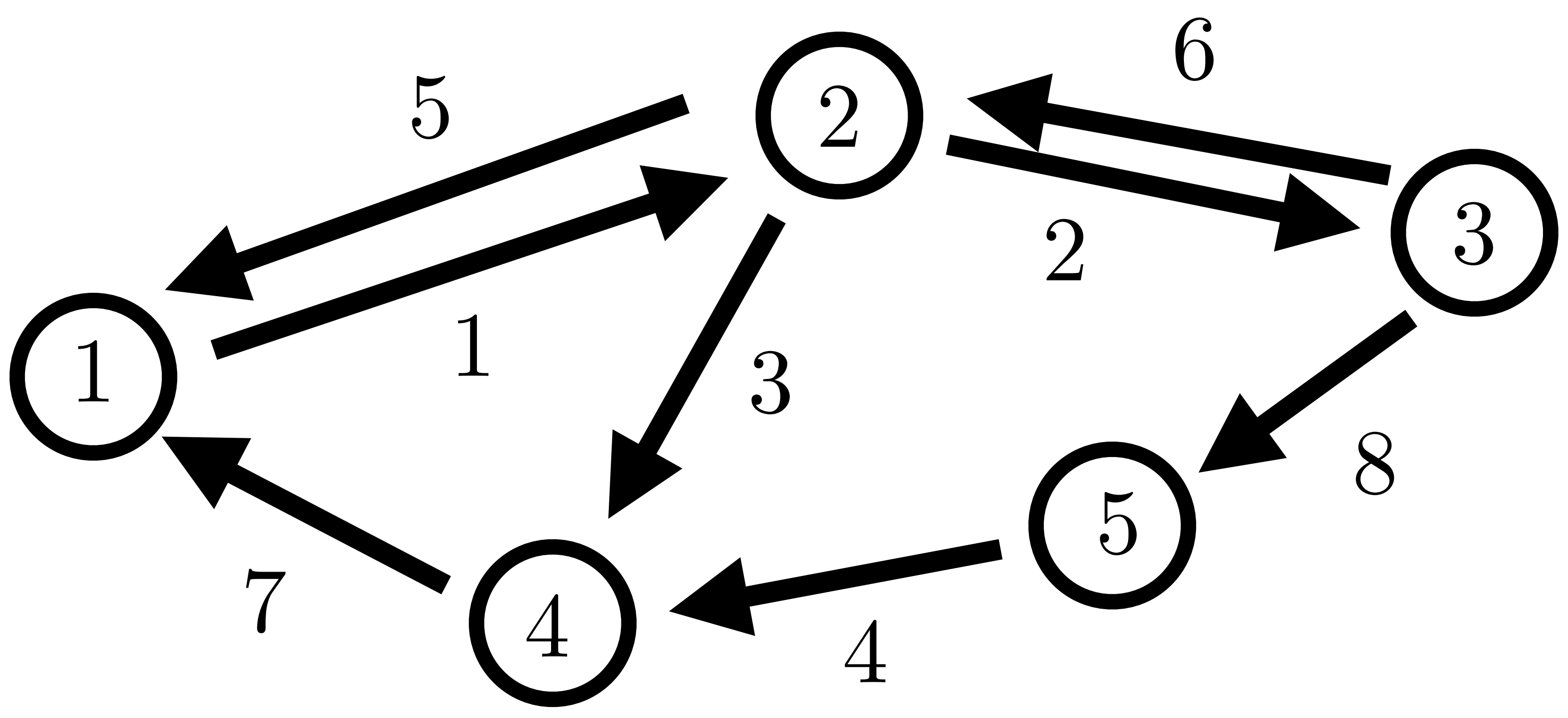

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"heights" of nodes

Linear ODE $\quad \dot{u} = -Lu$

Eigenvectors are oscillation modes

"Vibration modes" of a graph

Graph Laplacians

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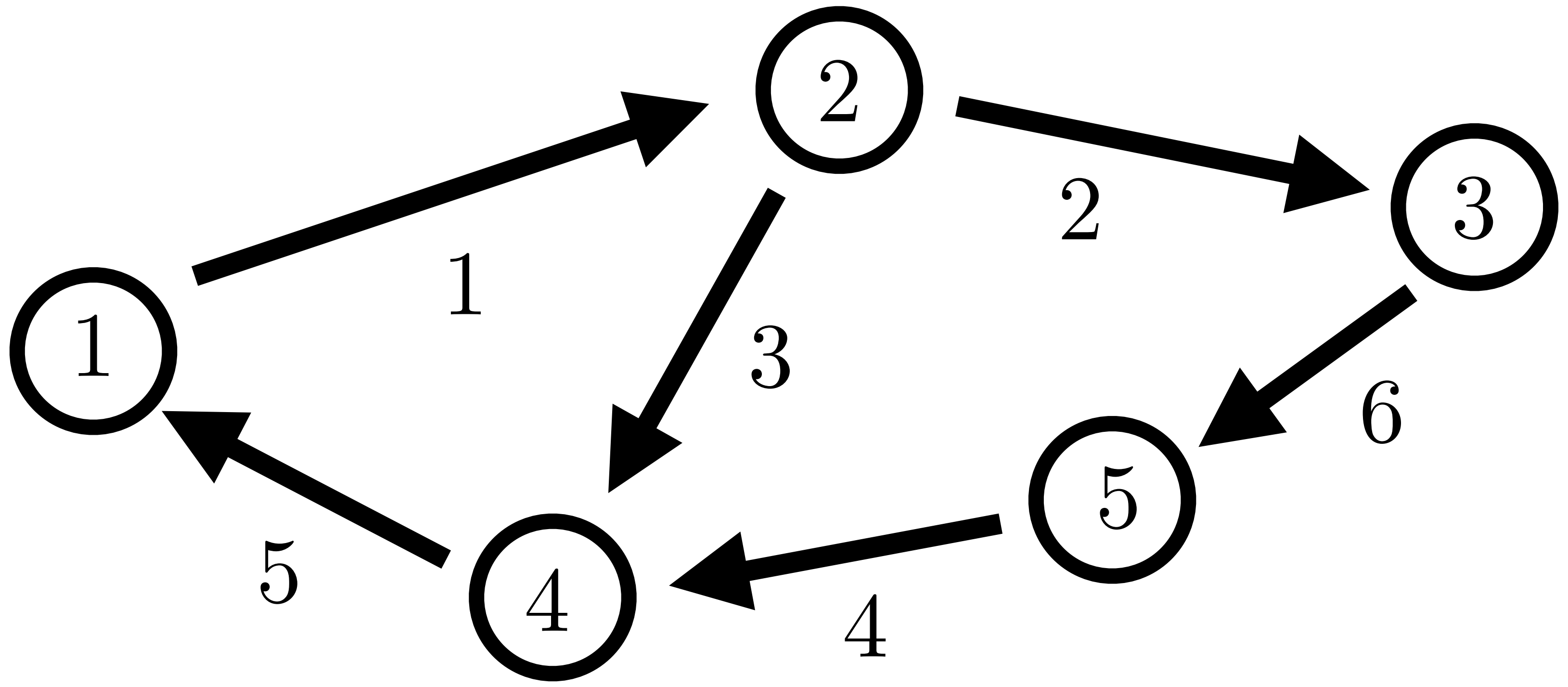
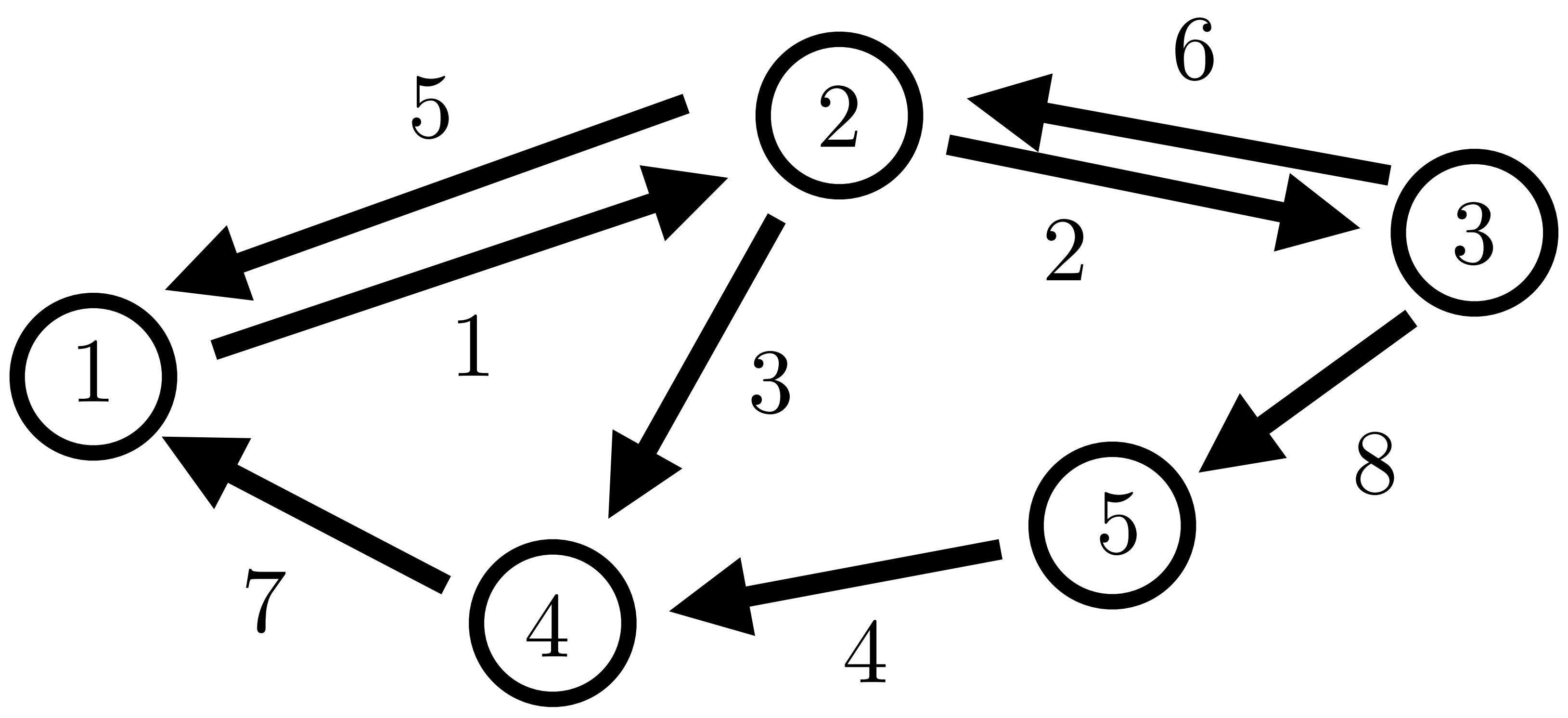


Laplacian $\quad L = DD^T$

$$L = DD^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Graph Laplacians



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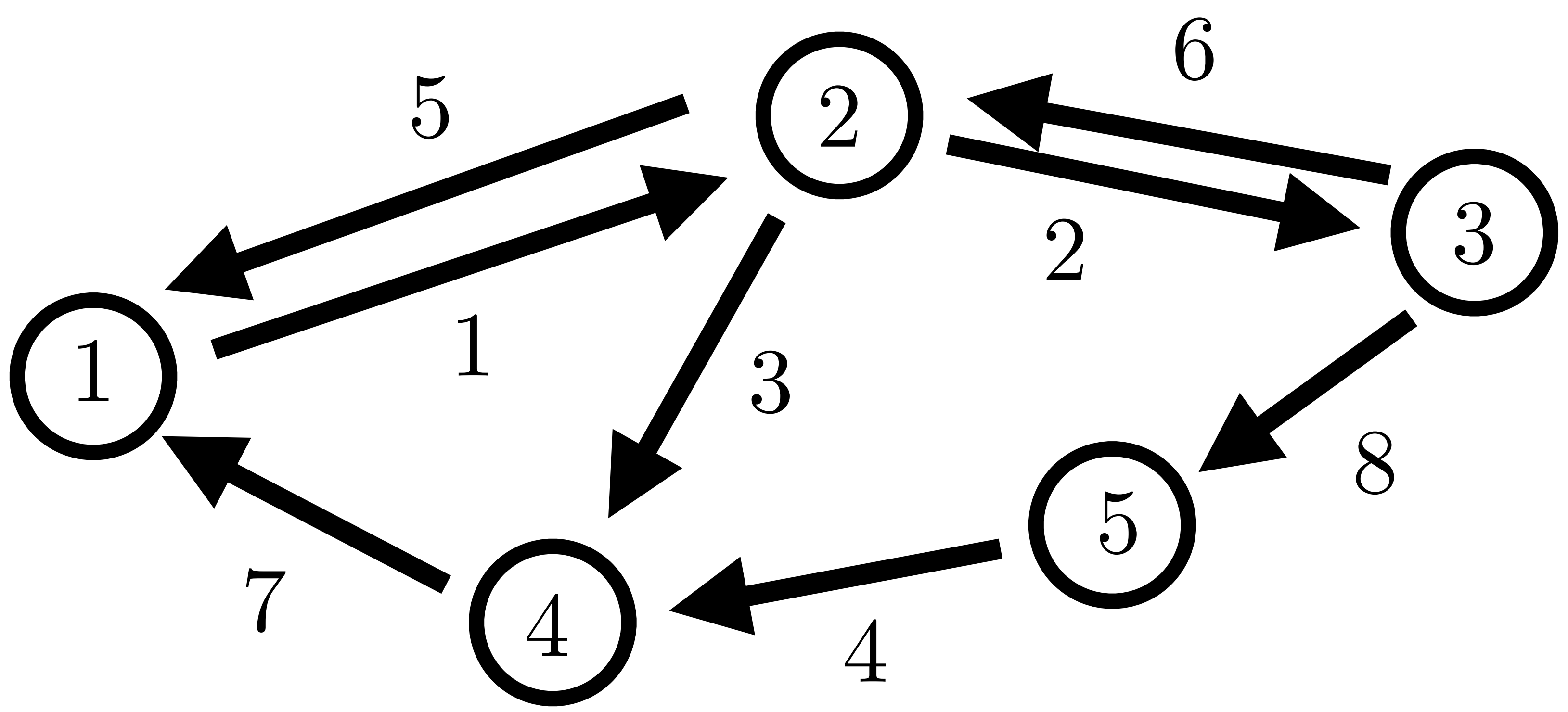

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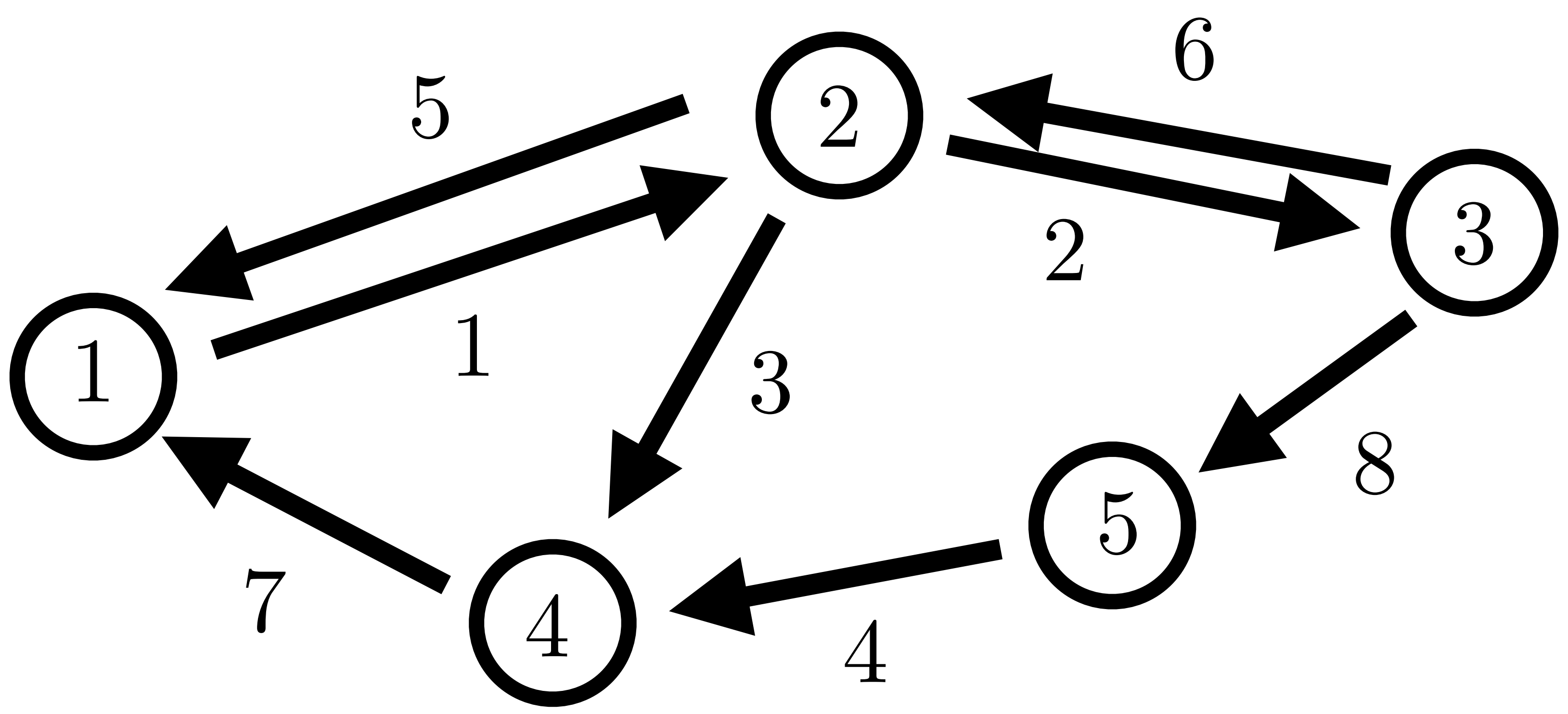

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Eigenvalues $\quad \underbrace{0 = \cdots = 0}_{\text{num of connected components}} < \lambda_1 \leq \cdots \leq \lambda_n$

Graph Laplacians

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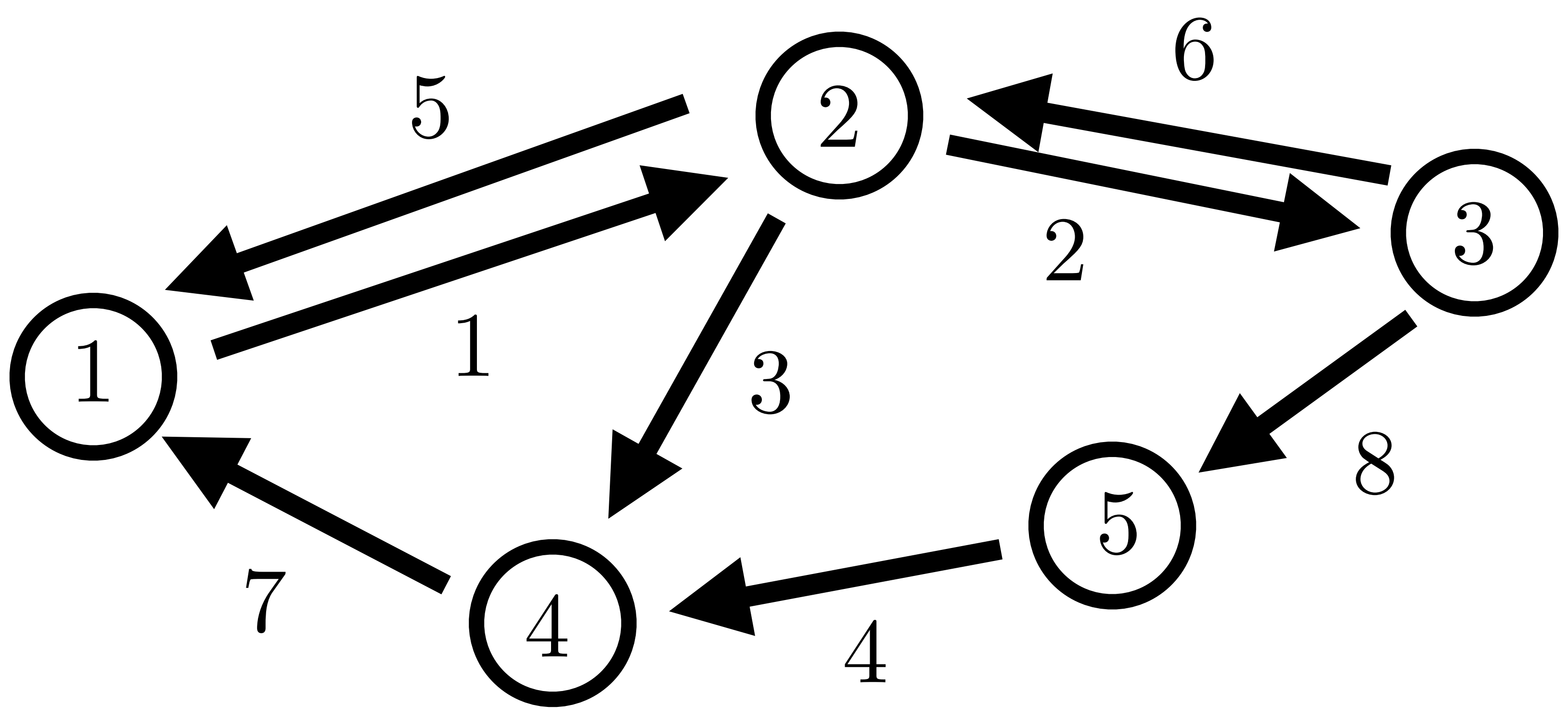

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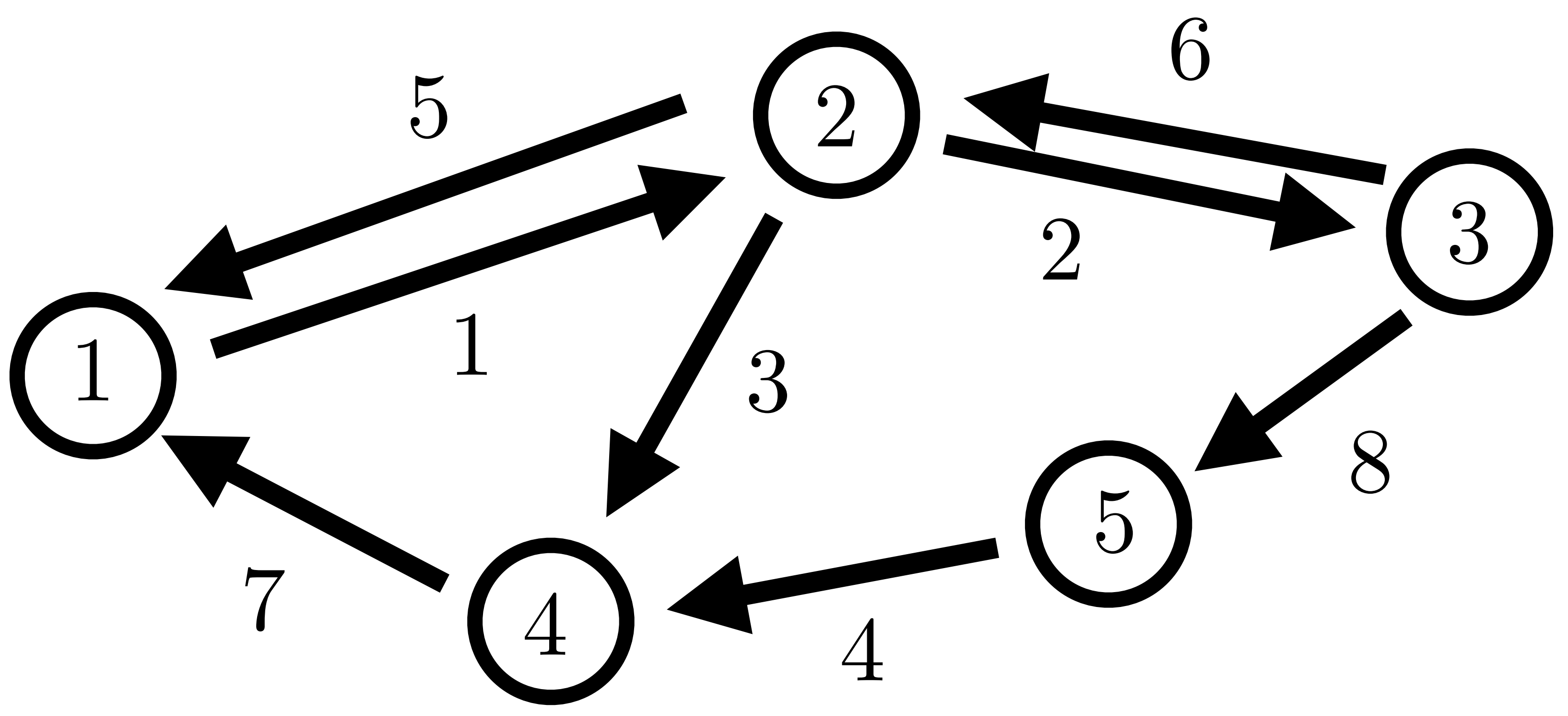
Eigenvectors

Constant vectors
(zero eigenvalues)

$$\longleftarrow \begin{bmatrix} \boxed{\begin{matrix} | \\ \bar{\mathbf{1}} \\ | \end{matrix}} & \begin{matrix} | \\ U_1 \\ | \end{matrix} & \cdots & \begin{matrix} | \\ U_k \\ | \end{matrix} \end{bmatrix}$$

Graph Laplacians



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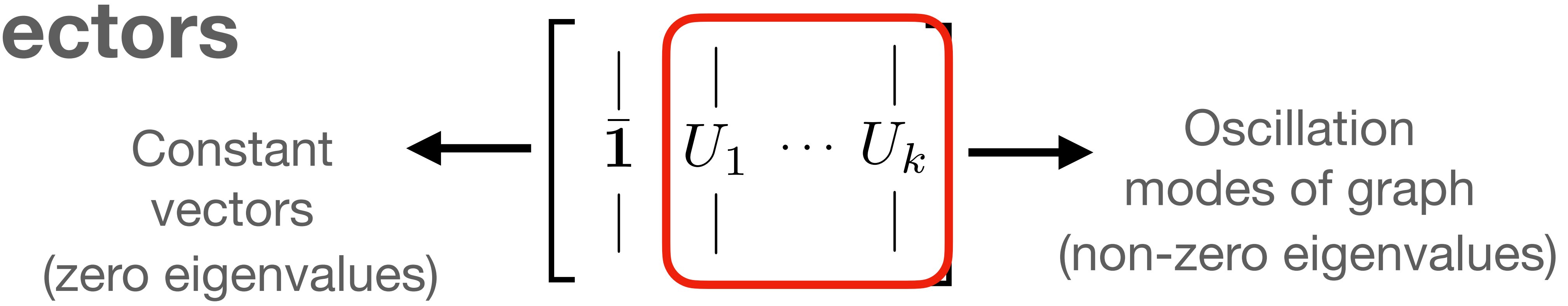
Eigenvectors

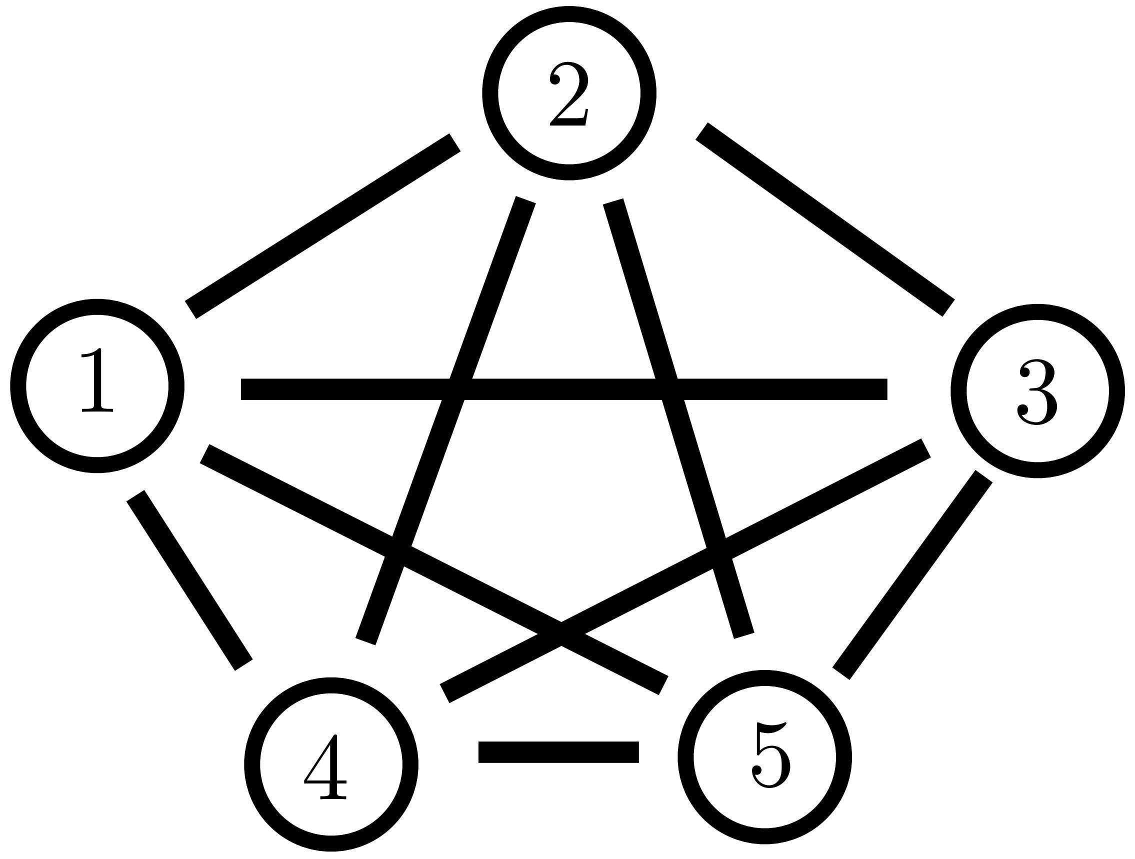


Constant vectors (zero eigenvalues)

$$\begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix}$$

Oscillation modes of graph (non-zero eigenvalues)

Graph Laplacians



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Laplacian $\quad L = DD^T \quad$ **Complete Graph**

$$L = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

Eigenvalues $\quad 0 < |\mathcal{V}| = \cdots = |\mathcal{V}|$

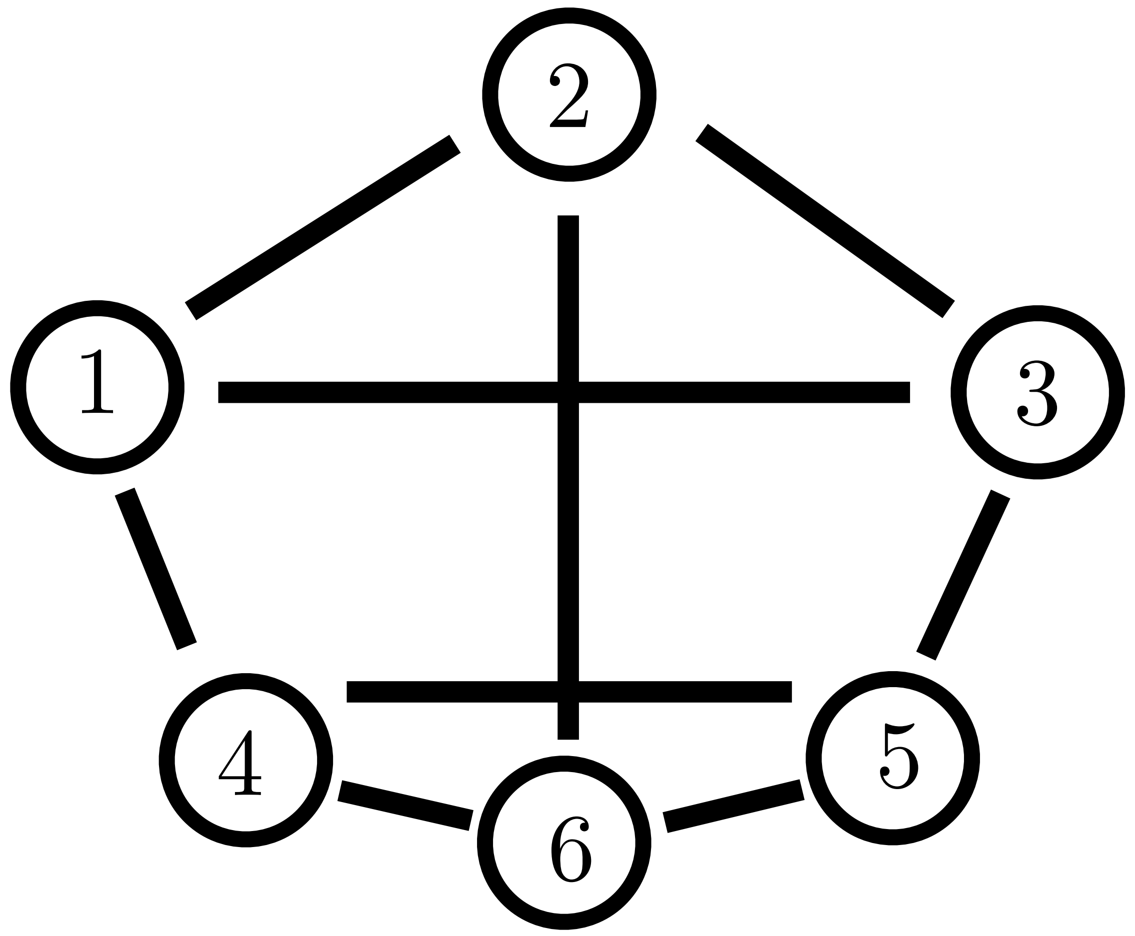
Eigenvectors

$$\begin{bmatrix} | & \boxed{\begin{matrix} | & & | \\ U_1 & \cdots & U_k \\ | & & | \end{matrix}} \\ \mathbf{1} & \\ | & \end{bmatrix} \longrightarrow$$

Any orthonormal basis vectors perpendicular to $\mathbf{1}$

Proof (sketch) $\quad L = -\mathbf{1}\mathbf{1}^T + |\mathcal{V}| I$

Graph Laplacians



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Laplacian $\quad L = DD^T$

d-Regular Graph

(all nodes have same degree)

$$L = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

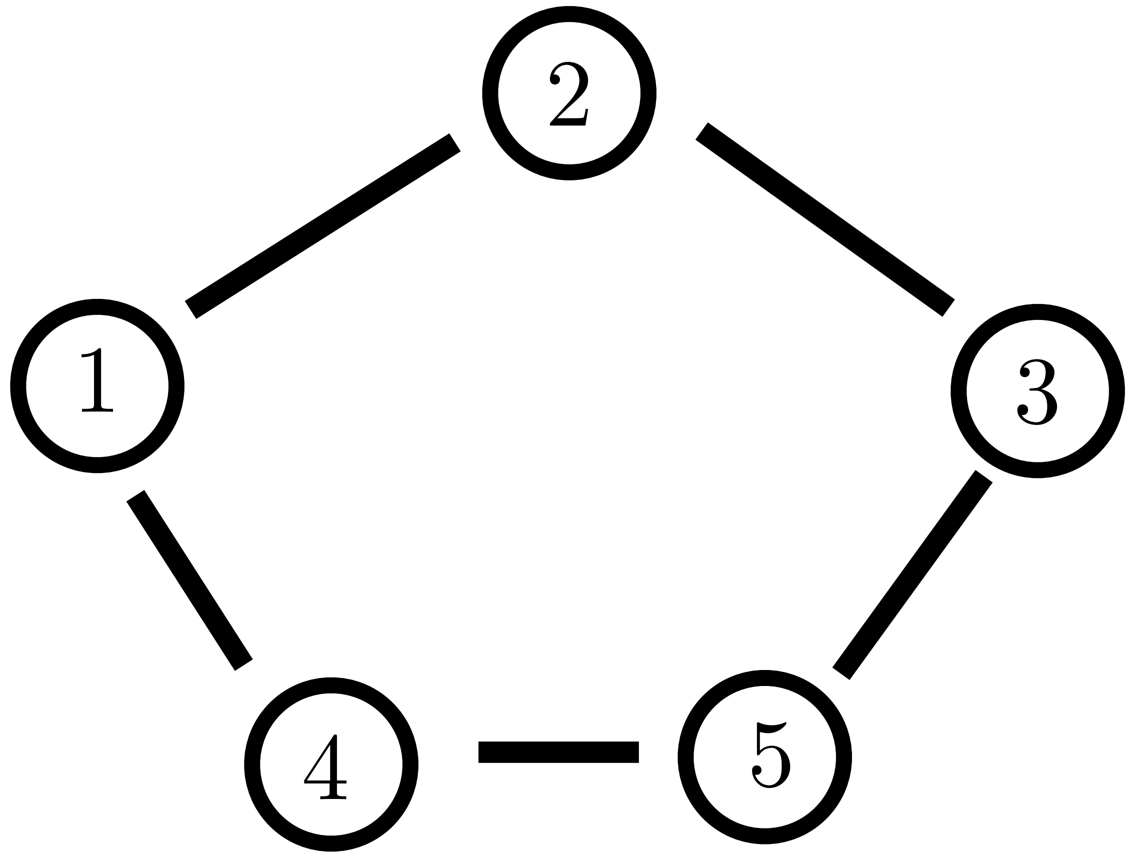
Eigenvalues (same as - adjacency matrix + d)

Eigenvectors (same as adjacency matrix)

see following slides

Proof (sketch) $\quad L = \Delta - A = dI - A$

Graph Laplacians



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Laplacian $\quad L = DD^T$

Cycle Graph

(or any circulant graph)

$$L = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{1}} & U_1 & \cdots & U_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & - & 0 & - \\ | & \lambda_1 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ | & 0 & \cdots & \lambda_k \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{1}}^T & - \\ - & U_1^T & - \\ & \vdots & \\ - & U_k^T & - \end{bmatrix}$$

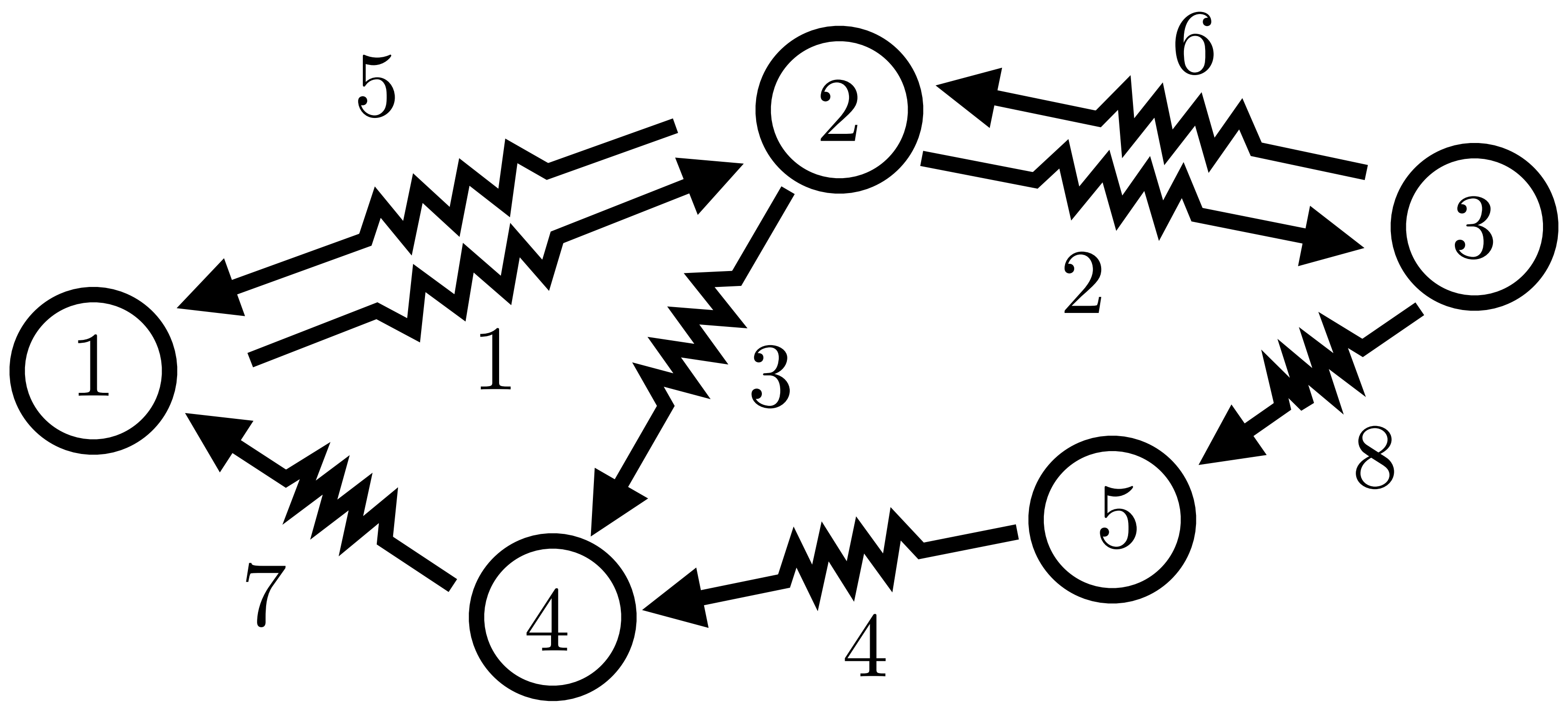
Eigenvalues (related to DFT)

Eigenvectors discrete Fourier basis vectors

Proof (sketch) Related to theory of circulant/shift matrices **Ask Dan (other materials)**

Note: Eigenvectors of L called Graph "Fourier" Transform …. extension of DFT

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

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Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Weighted Laplacian $\quad L_W = DWD^T$

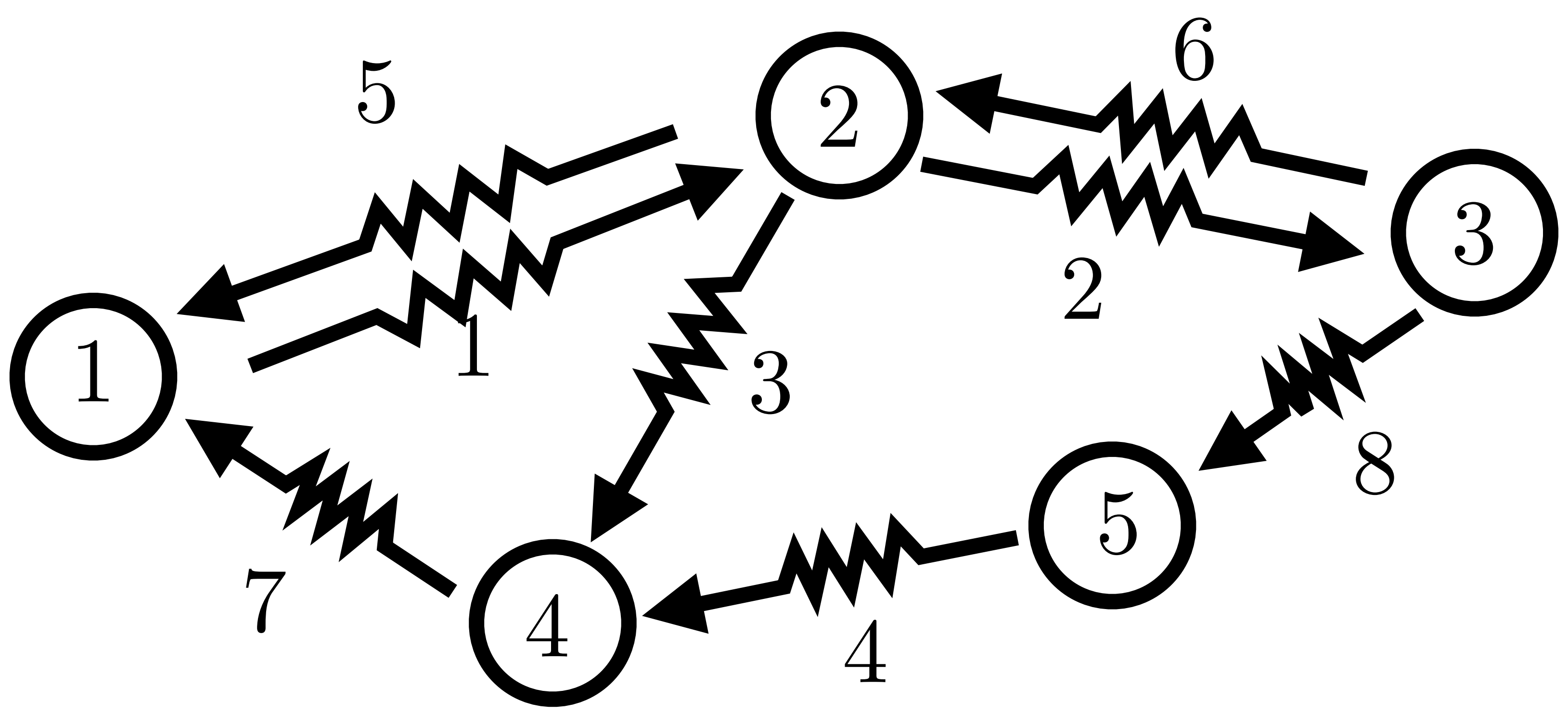
Edge weights $\quad W_e \geq 0 \qquad W = \operatorname{diag}\big([W_1 \ \cdots \ W_{|\mathcal{E}|}]\big)$

Interpretation: **resistance, travel time/cost**

$$L_W = DWD^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

$$= \begin{bmatrix} | & | \\ U'_W & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {U'}_W^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}^T$$

Graph Laplacians



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Weighted Laplacian $L_W = DWD^T$

Action: $$L_W u = \underbrace{\Big[\ D\ \Big]\underbrace{\Big[\ W\ \Big]\Big[\ D^T\ \Big]\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges scaled by weights}}}_{\text{... summed resulting tension on nodes}}$$

“heights” of nodes

Graph Laplacians

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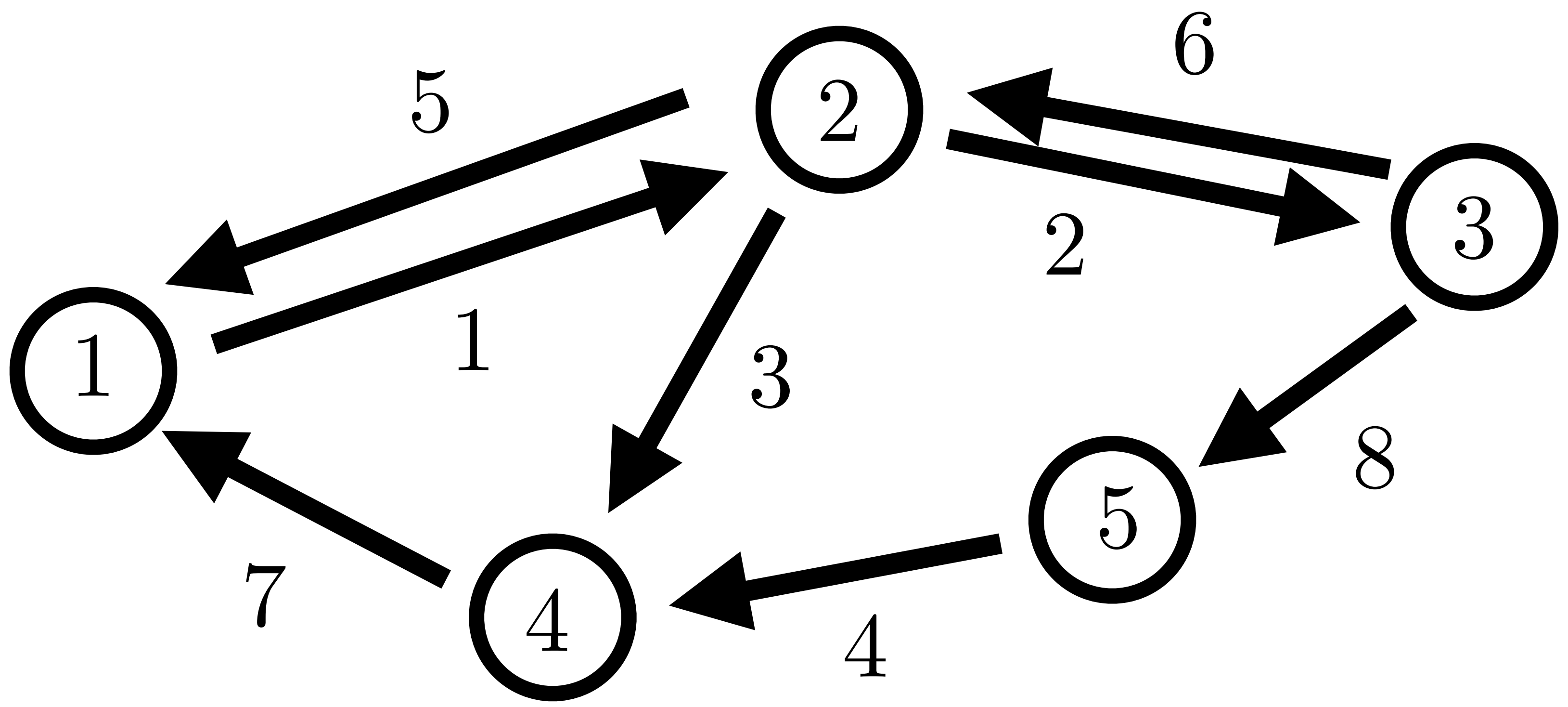

Edge Laplacian $\quad L_e = D^T D$

Graph Laplacians

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$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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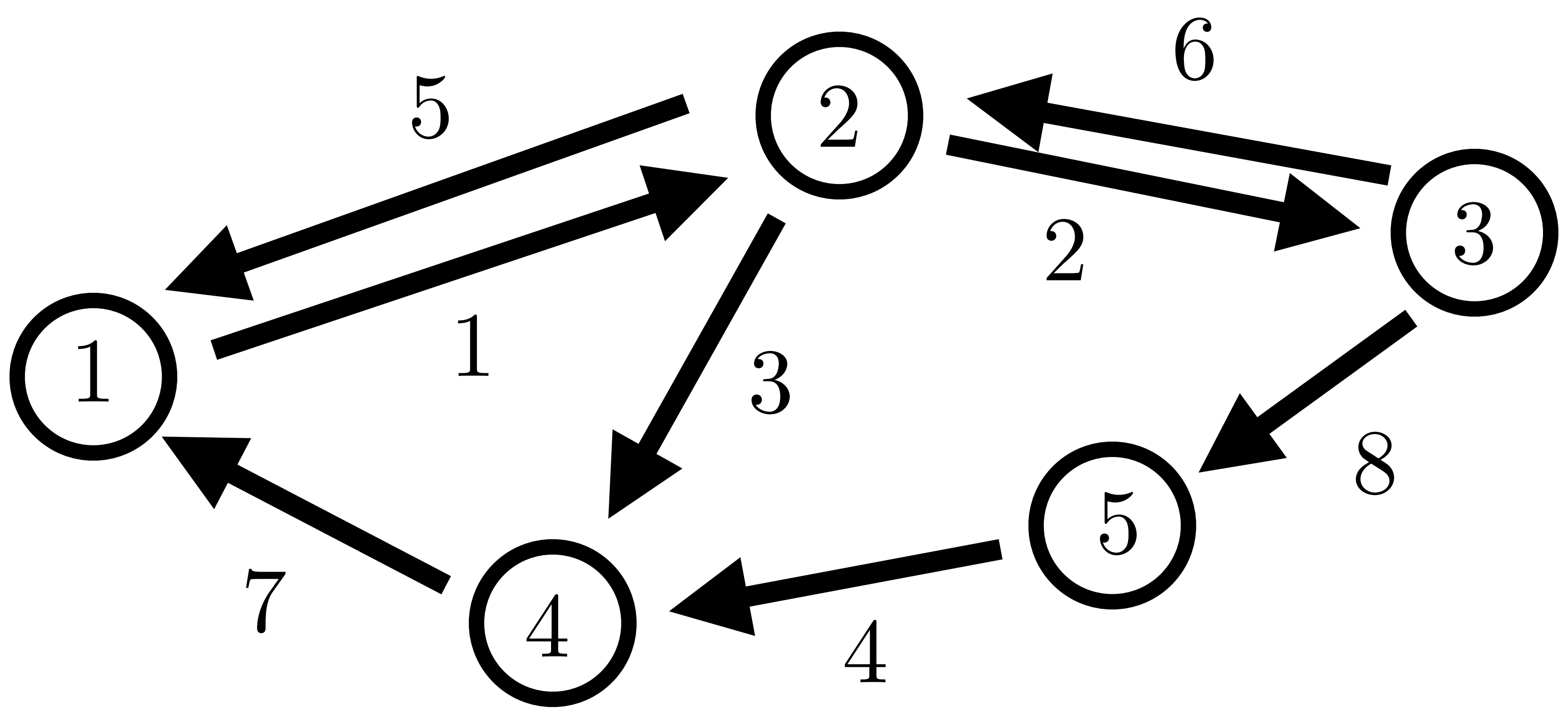


Edge Laplacian $\quad L_e = D^T D$

Action:

$$L_e \tau = \underbrace{\left[\ D^T\ \right] \underbrace{\left[\ D\ \right] \begin{bmatrix} | \\ \tau \\ | \end{bmatrix}}_{\text{...summed tension on nodes}}}_{\text{... differential in tension along edges}}$$

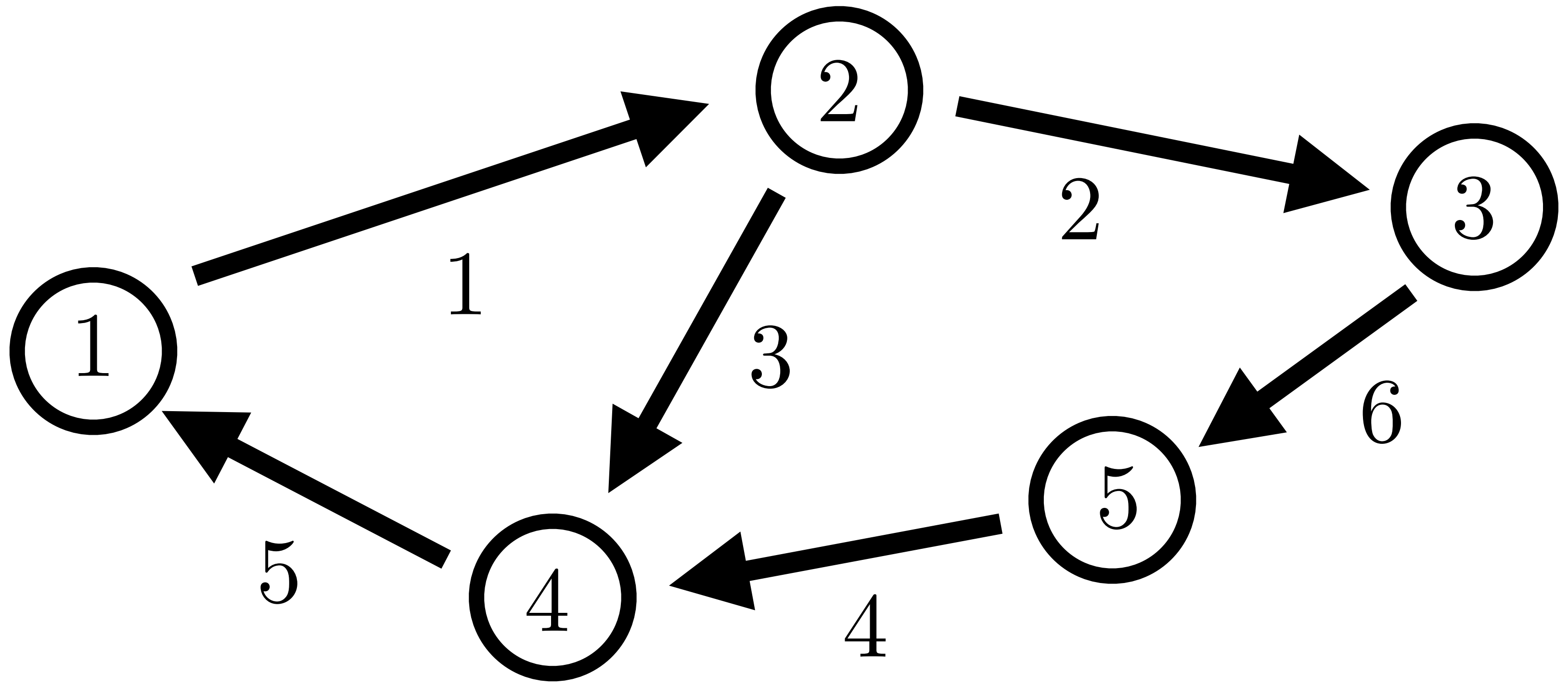
"Tension" in edges

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

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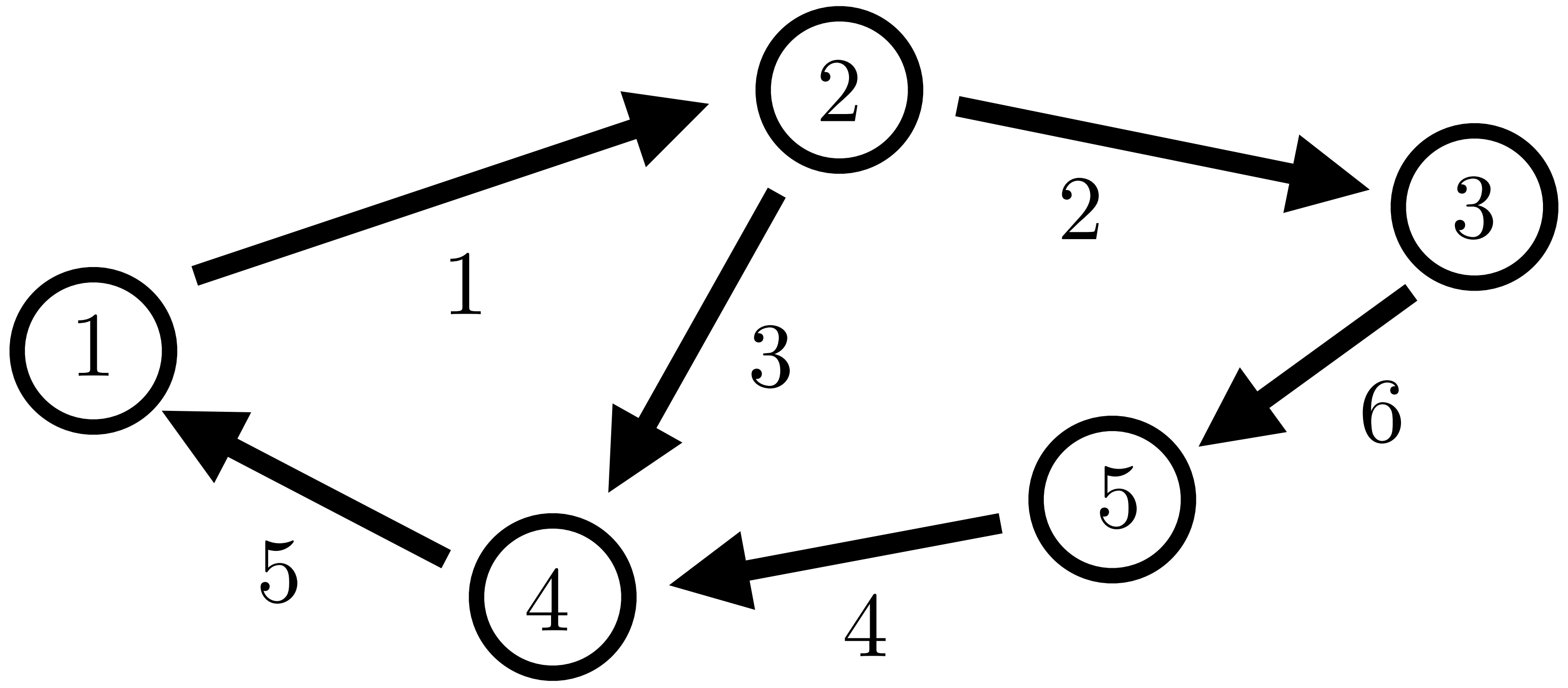
Degree & Adjacency Matrix



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$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

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$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $L = DD^T$

Independent of edge direction

$$L = \Delta - A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Degree Matrix $\left[\ \Delta\ \right]_{vv} = |\mathcal{N}_v|$ **diagonal**

Adjacency Matrix $\left[\ A\ \right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Adjacency Matrix

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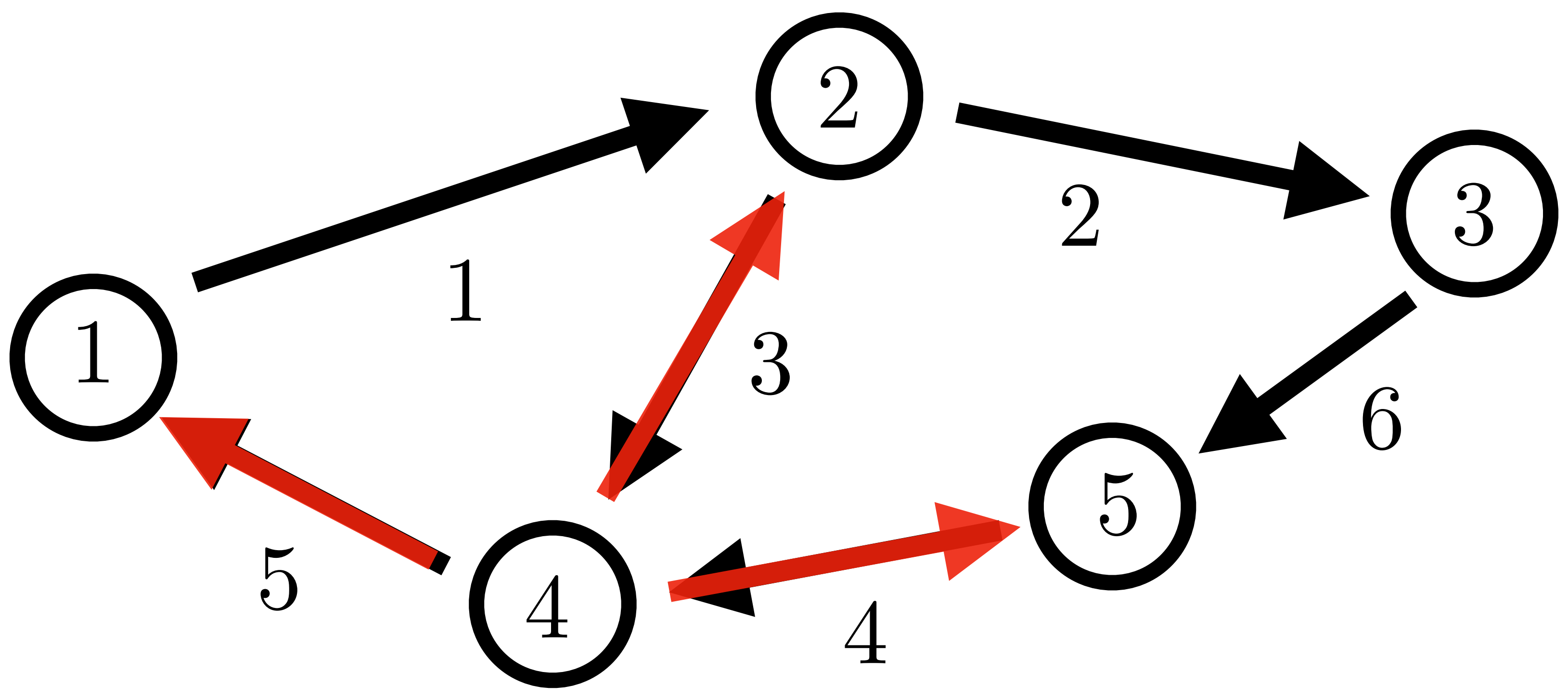

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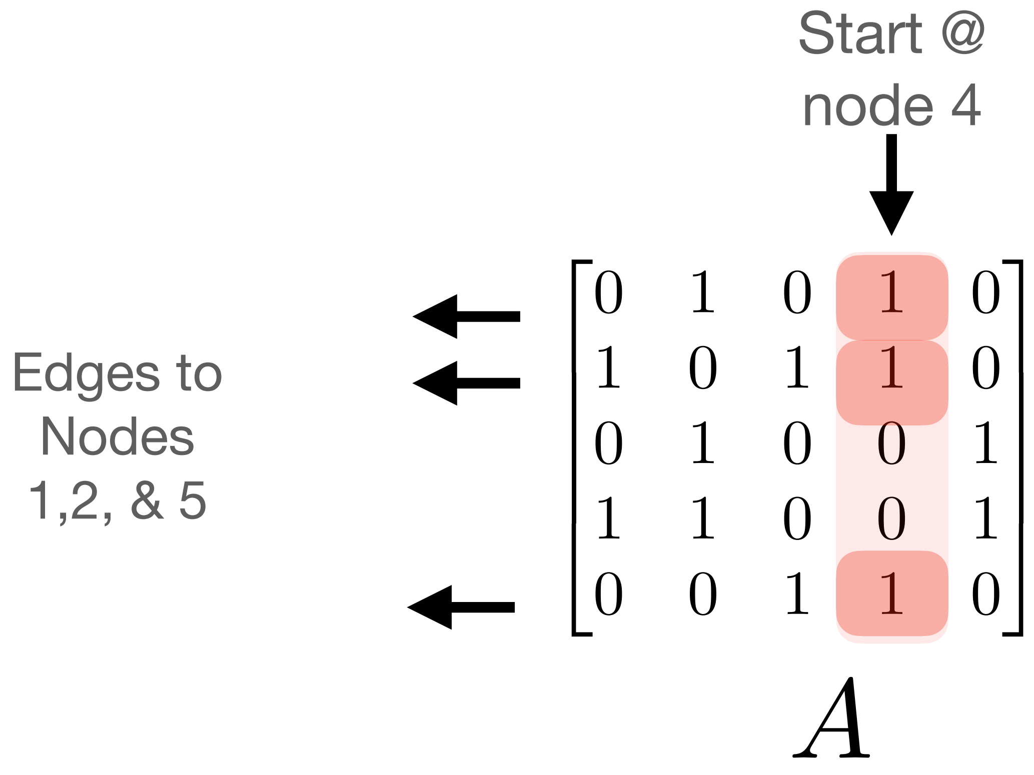
Adjacency Matrix



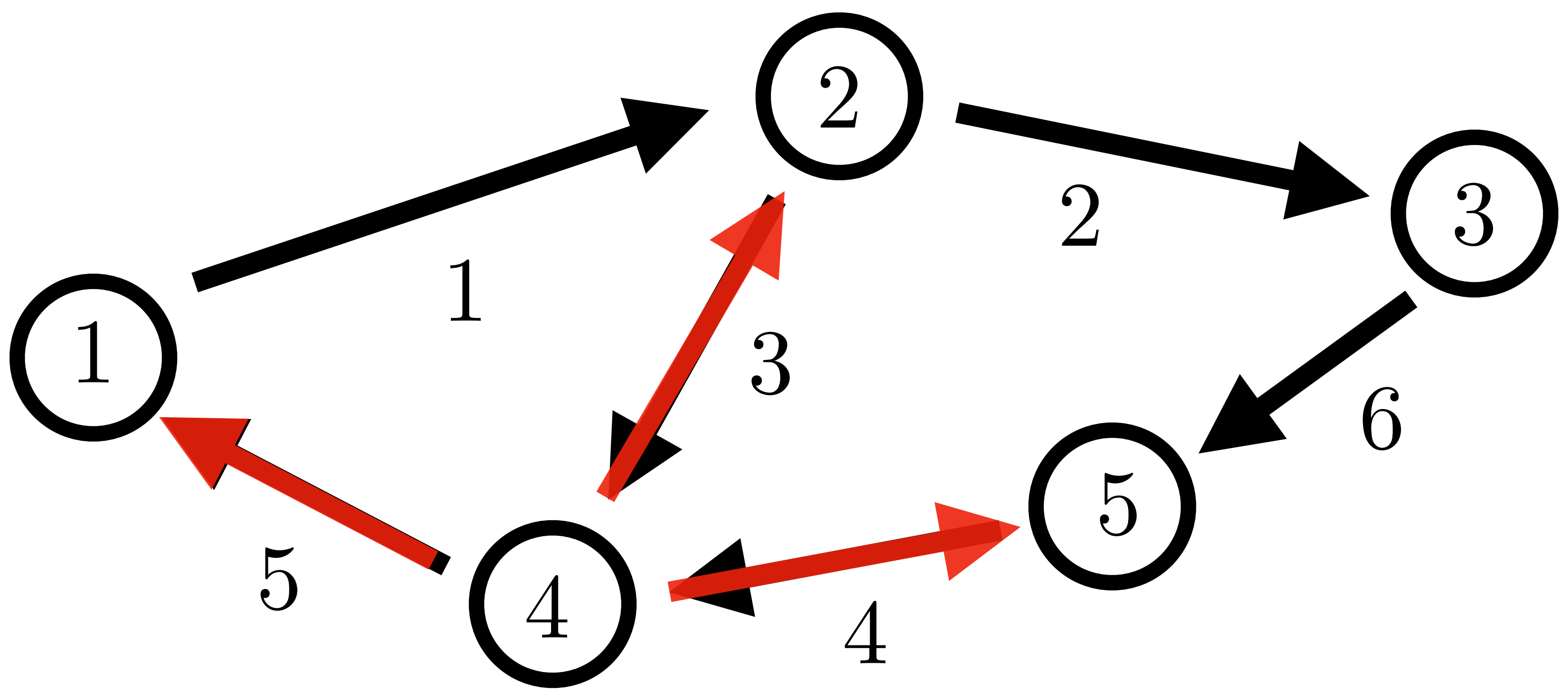
Start @ node 4

Edges to Nodes 1,2, & 5

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A

Adjacency Matrix



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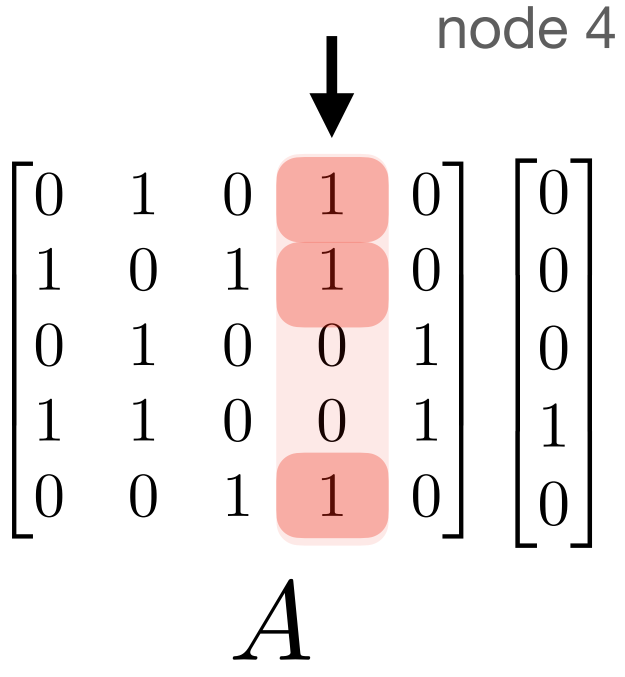
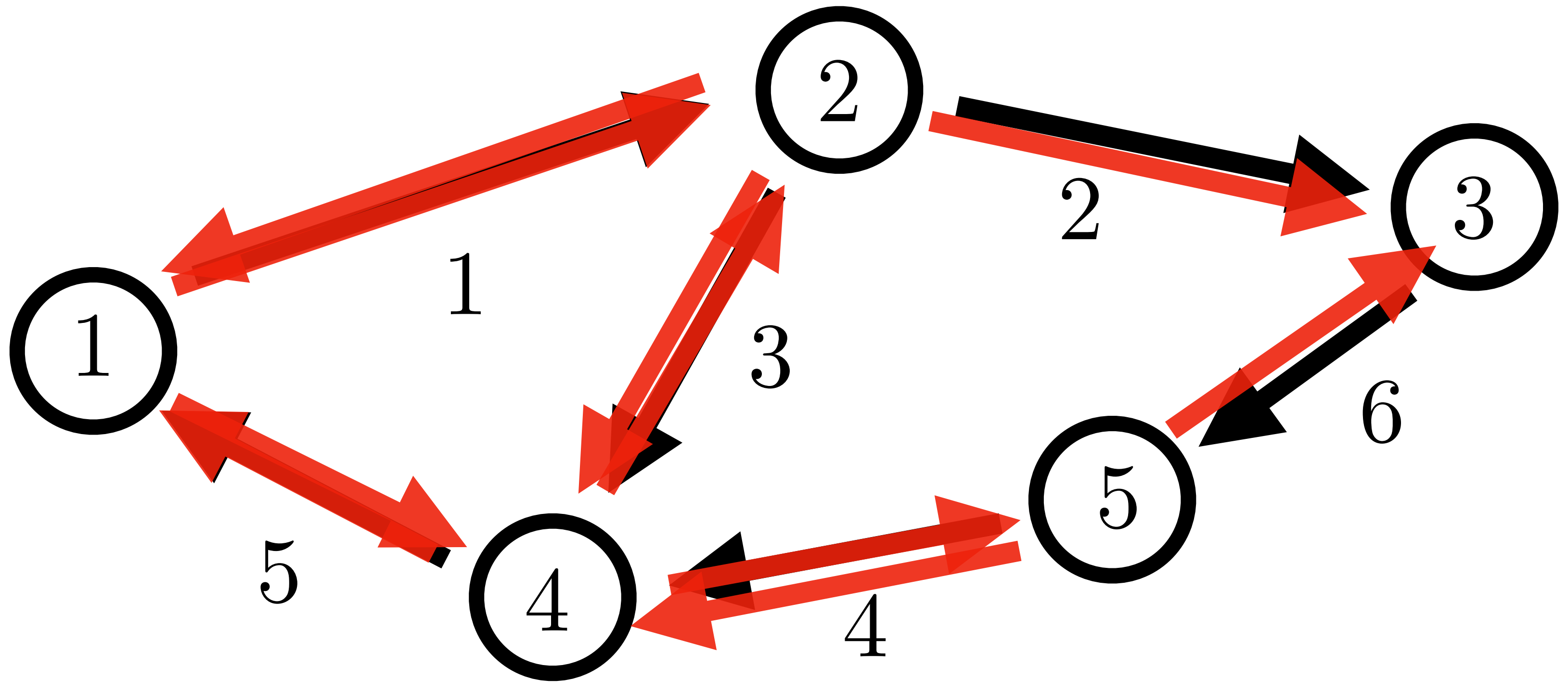
Powers of Adjacency

Start @ node 4



$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



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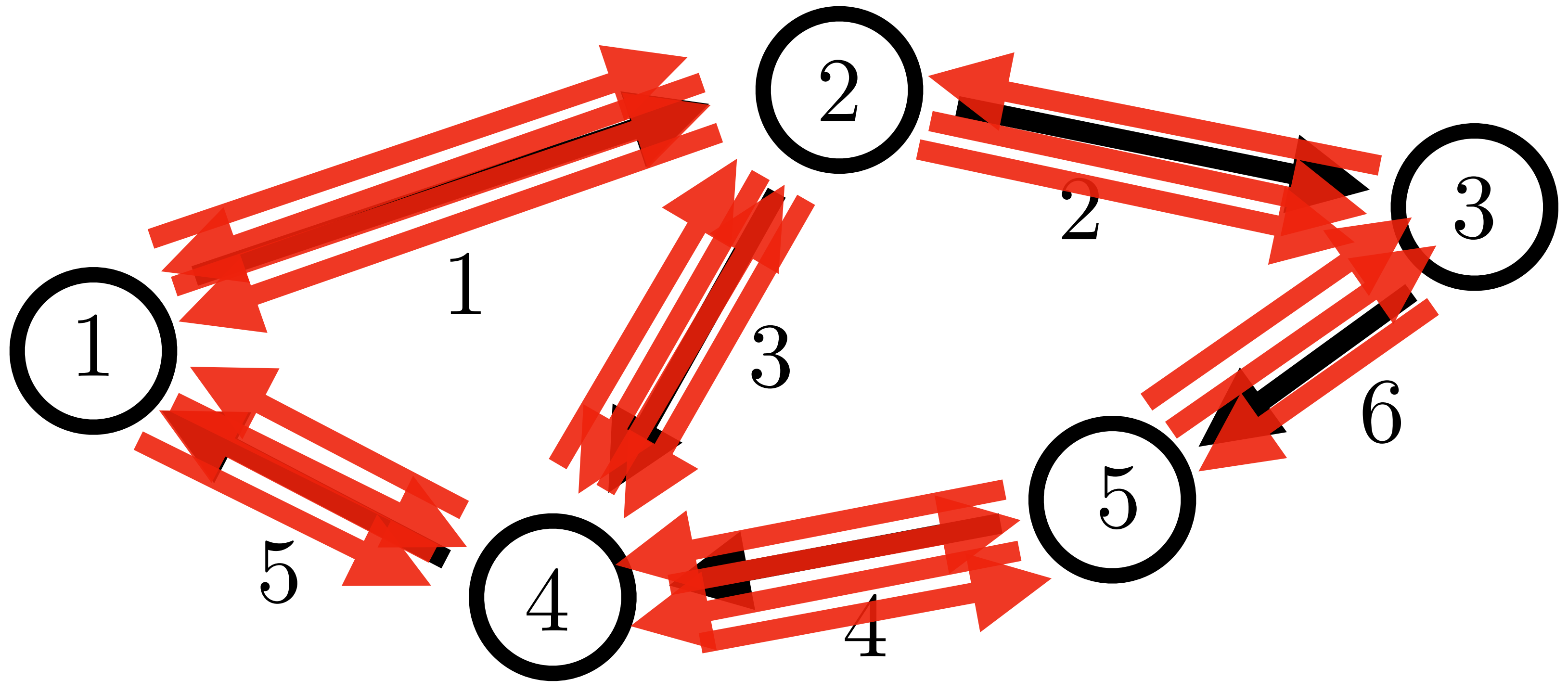
Powers of Adjacency

Start @ node 4

$$\underset{A}{\left[\begin{array}{ccccc} \text{Thru Node 1} & \text{Thru Node 2} & & & \text{Thru Node 5} \\ \downarrow & \downarrow & & & \downarrow \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]} \underset{A}{\left[\begin{array}{ccccc} & & & \downarrow & \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



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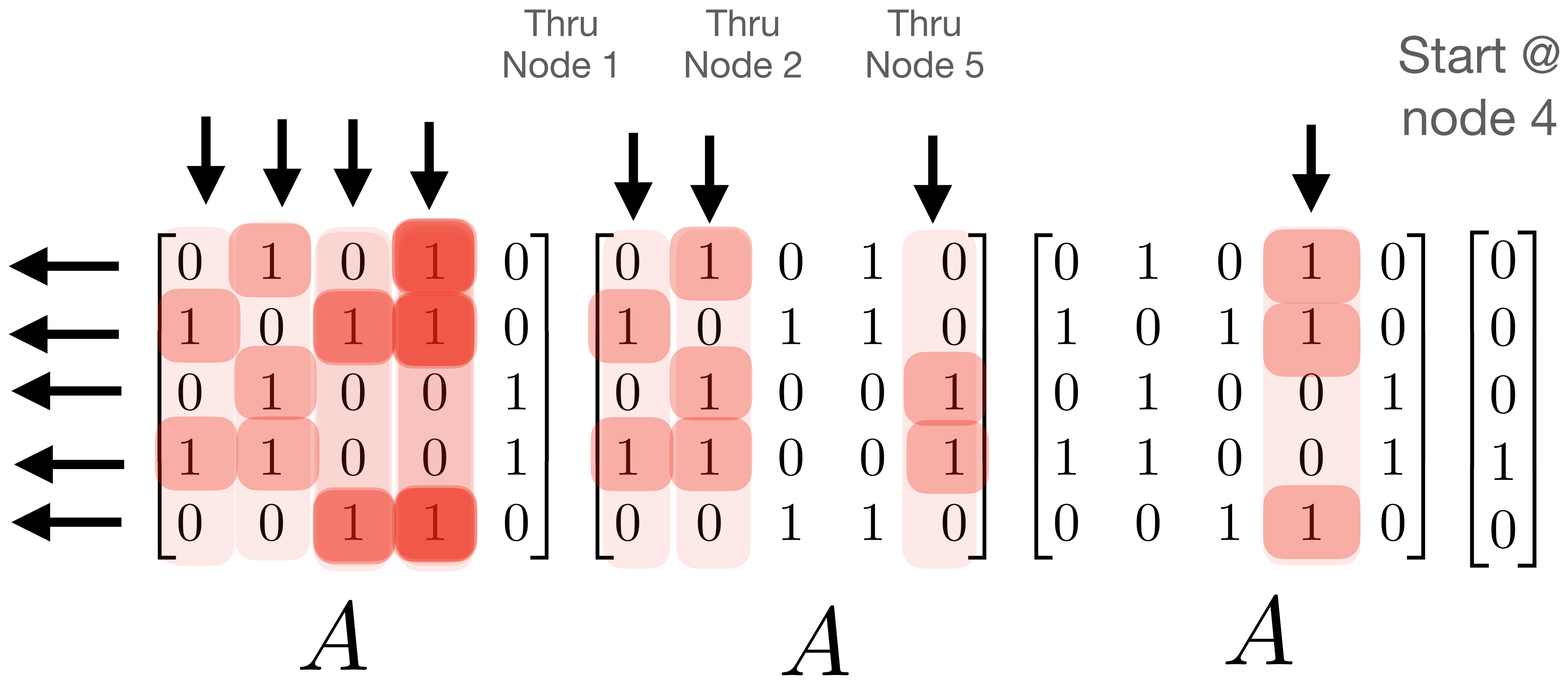
Powers of Adjacency



Thru Node 1

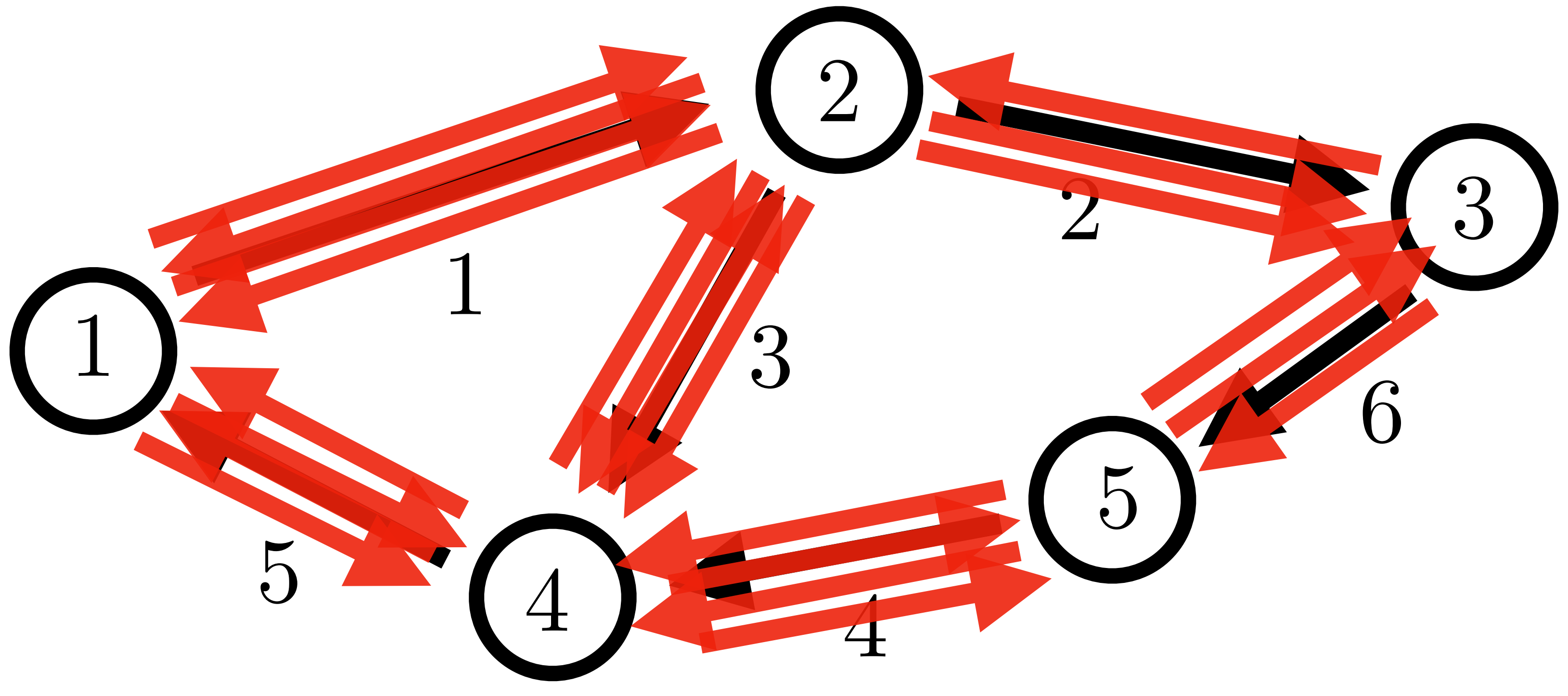
Thru Node 2

Thru Node 5

Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



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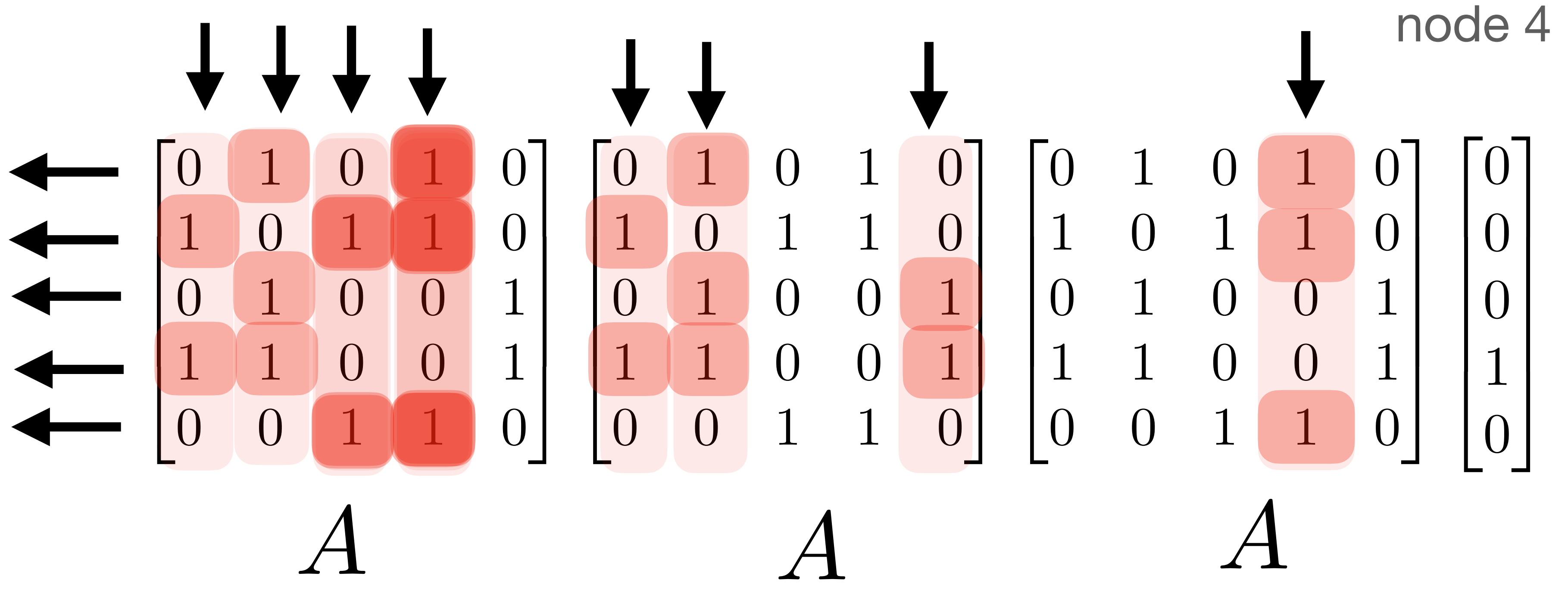
Powers of Adjacency



3-step paths from node 4 to node 1

3-step paths from node 4 to node 2

⋮

3-step paths from node 4 to node 5

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix

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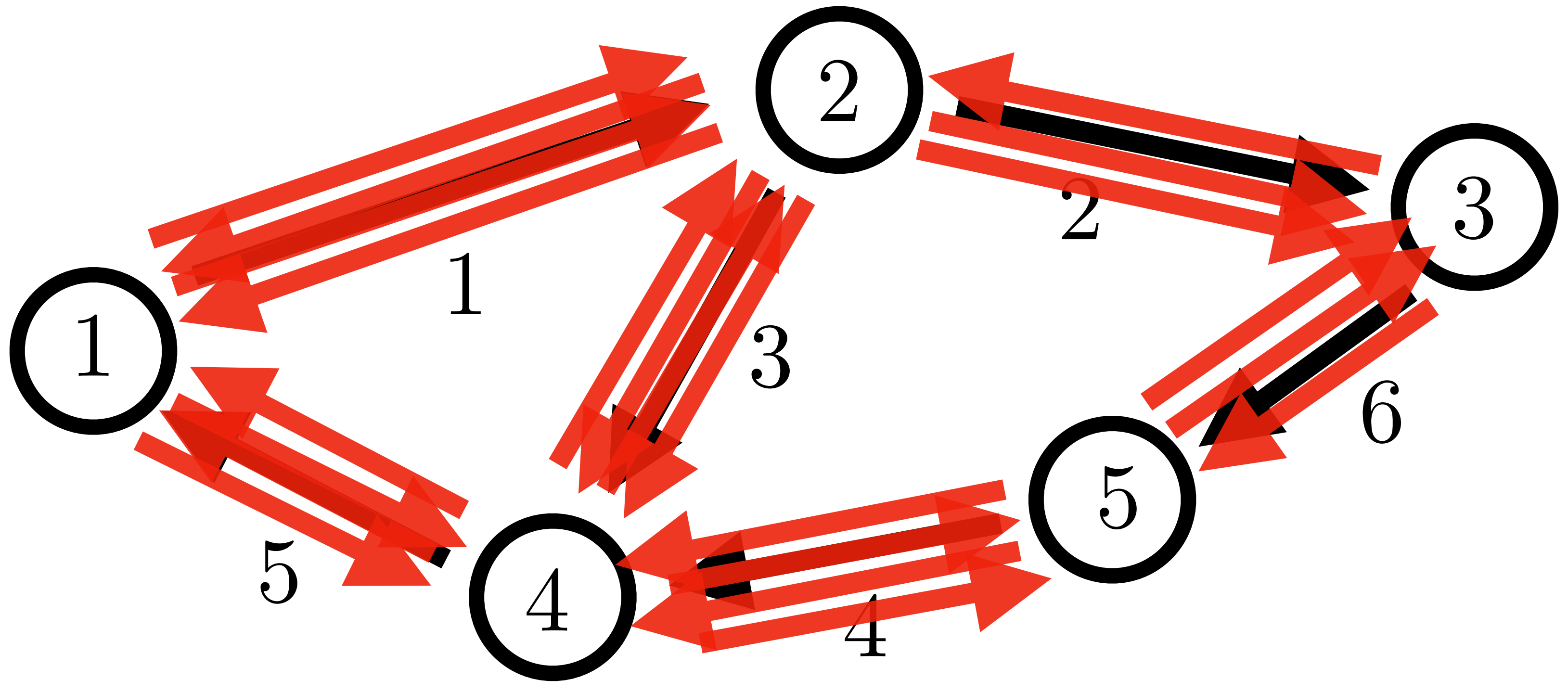
Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



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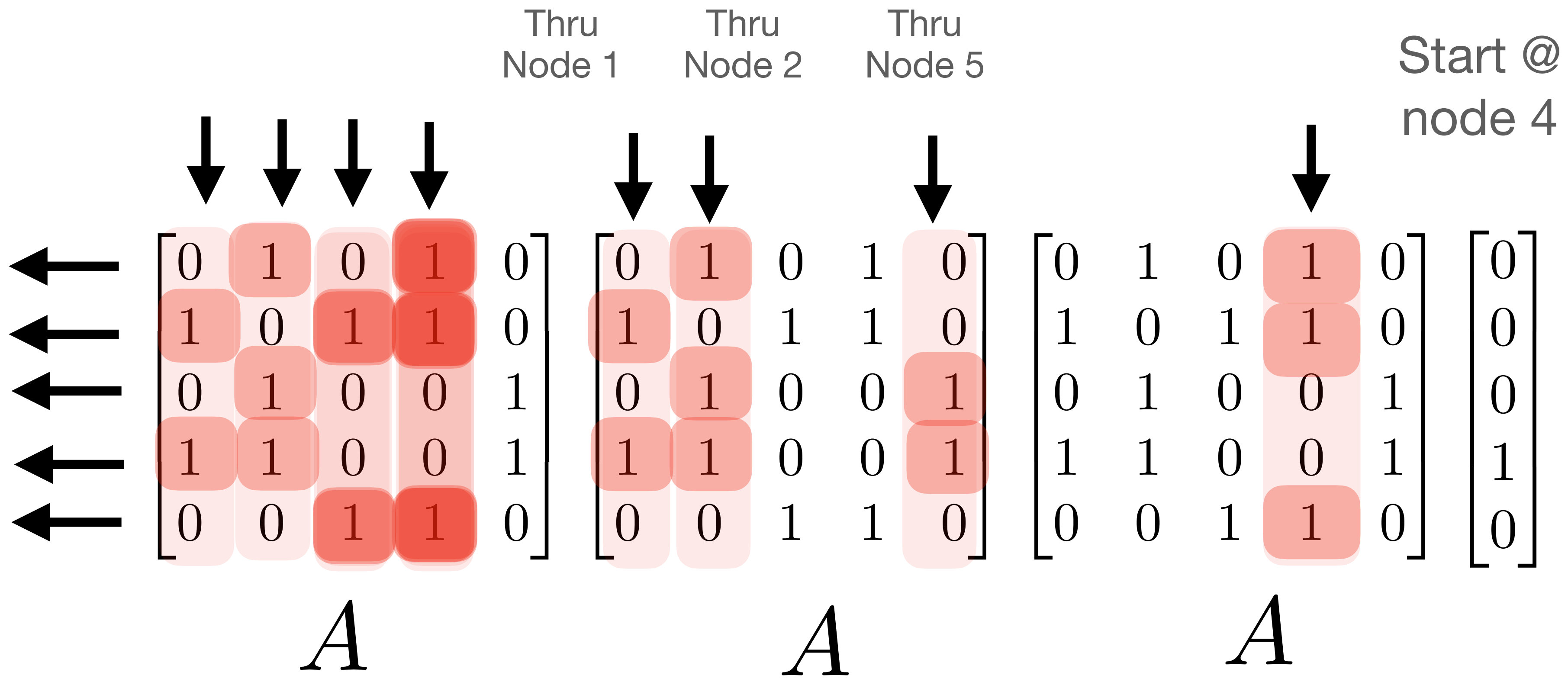
Powers of Adjacency

$$[\ A^k\ ]_{vv'}$$

 # k-step paths from node v to node v’



$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

REVIEW: Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} | & | \\ A_1 & A_2 \\ | & | \end{matrix}}_{A'} & \underbrace{\begin{matrix} | & | & | \\ A_3 & A_4 & A_5 \\ | & | & | \end{matrix}}_{A''} \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \end{bmatrix}$$

$$A'' = A'B$$

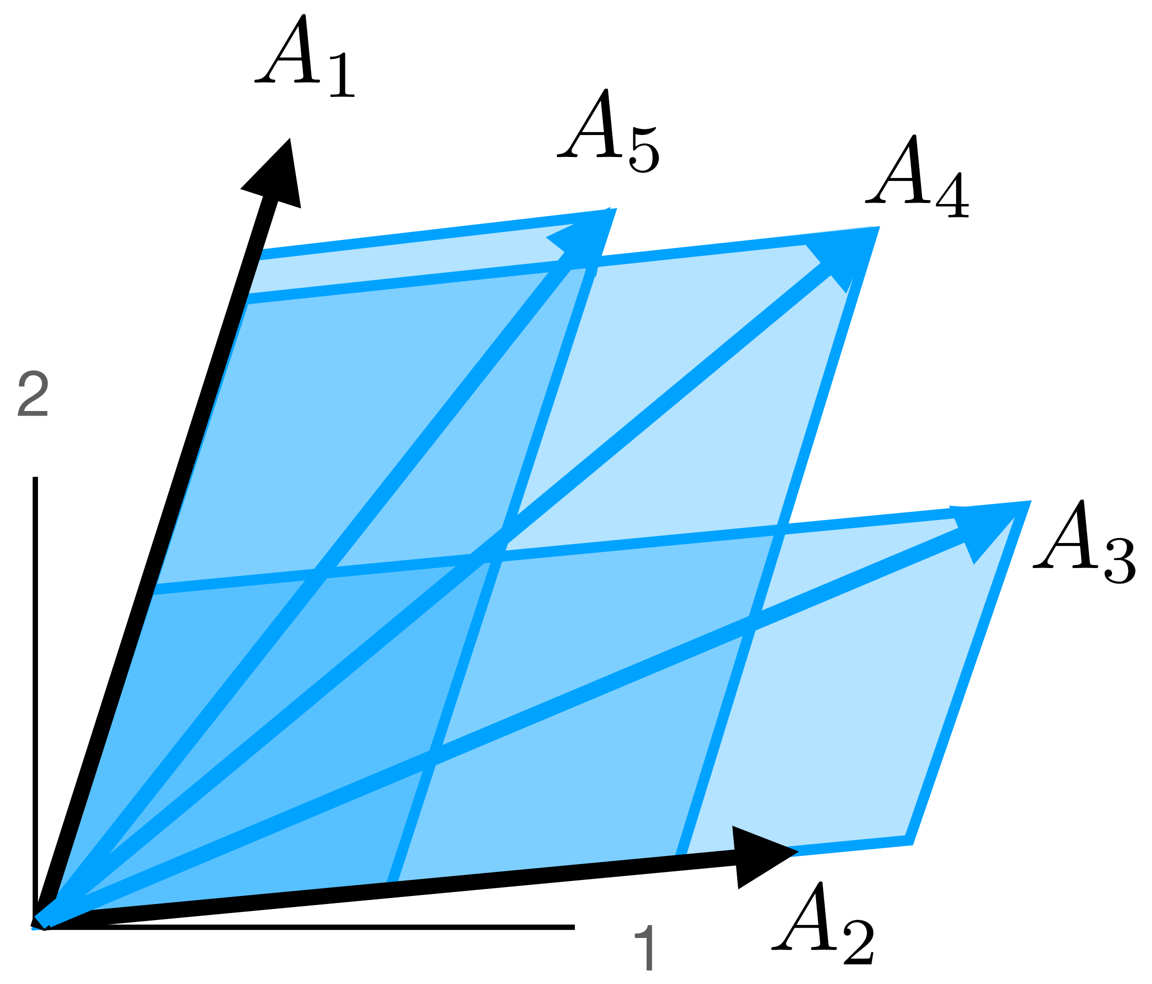
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} = A' \begin{bmatrix} I & B \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ A_3 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1B_{13} + A_2B_{23} & A_1B_{14} + A_2B_{24} & A_1B_{15} + A_2B_{25} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

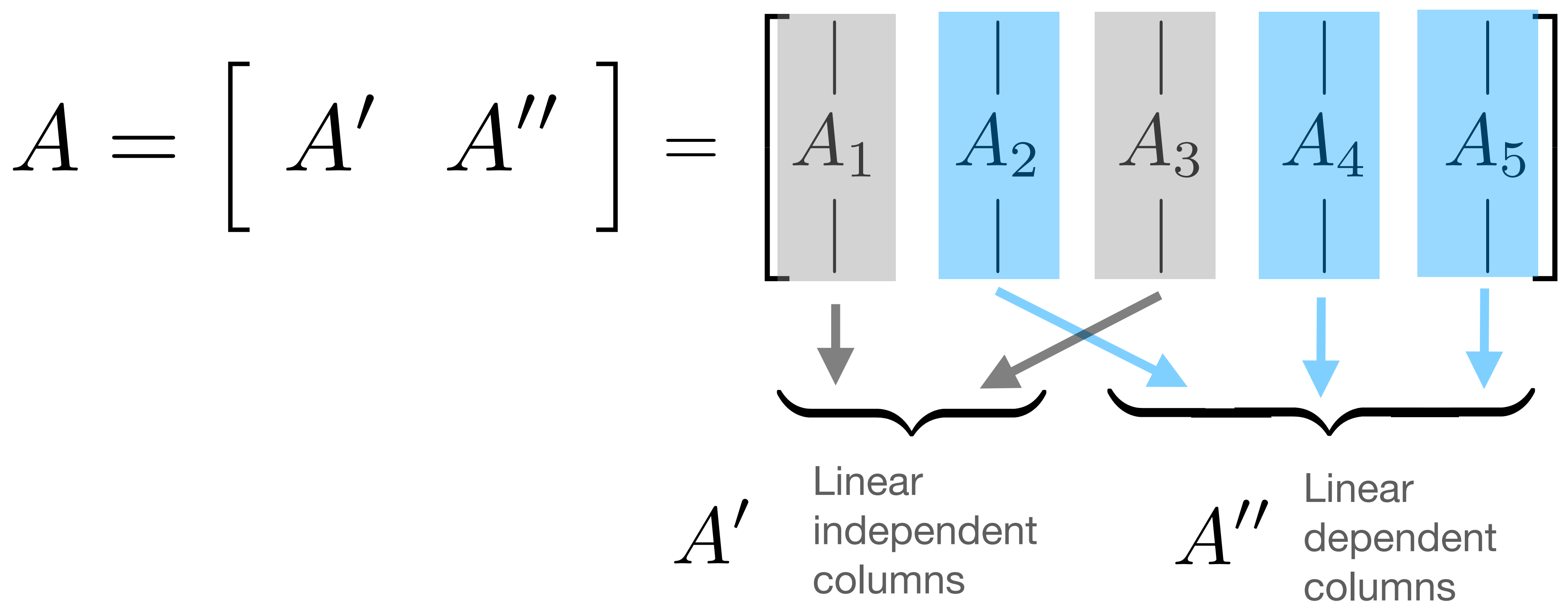
Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

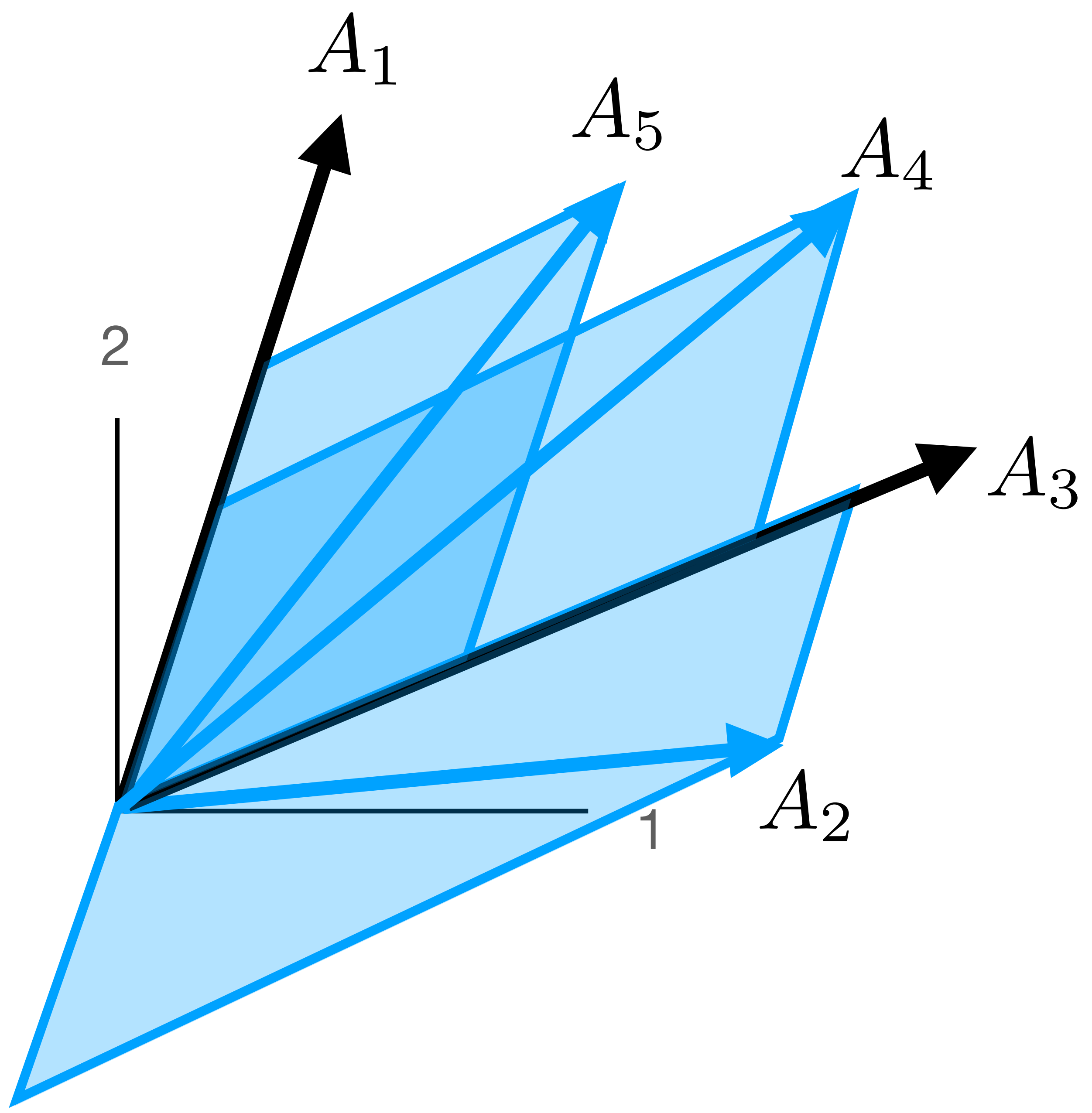
REVIEW: Nullspace - Column Geometry (Computation)



$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{33} & B_{34} & B_{35} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_3 B_{32} & A_1 B_{14} + A_3 B_{34} & A_1 B_{15} + A_3 B_{35} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ B_{33} & B_{34} & B_{35} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

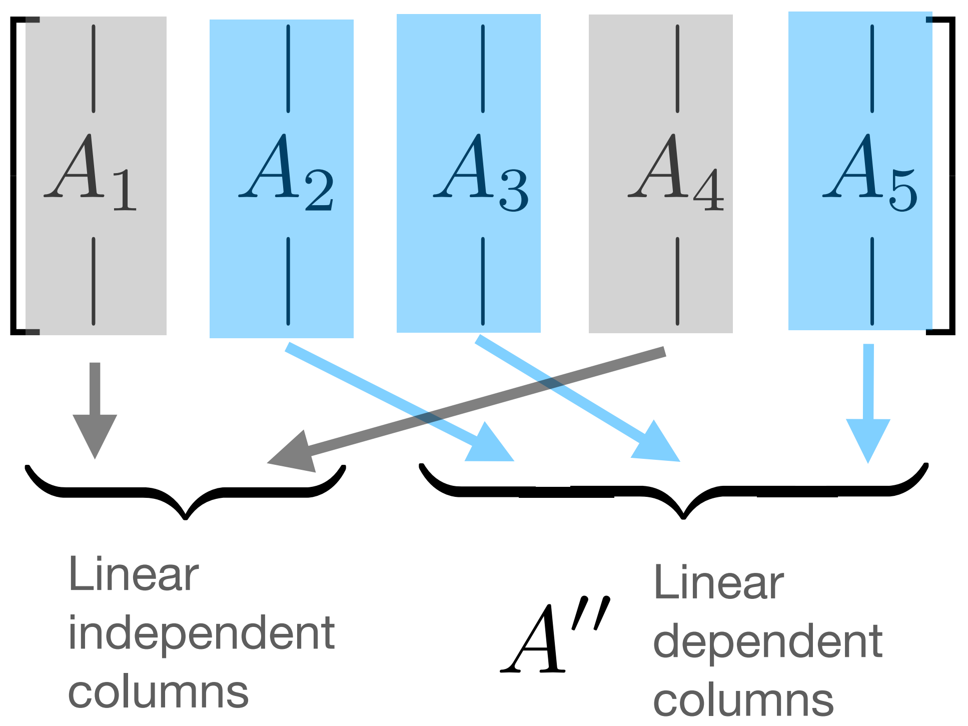
Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

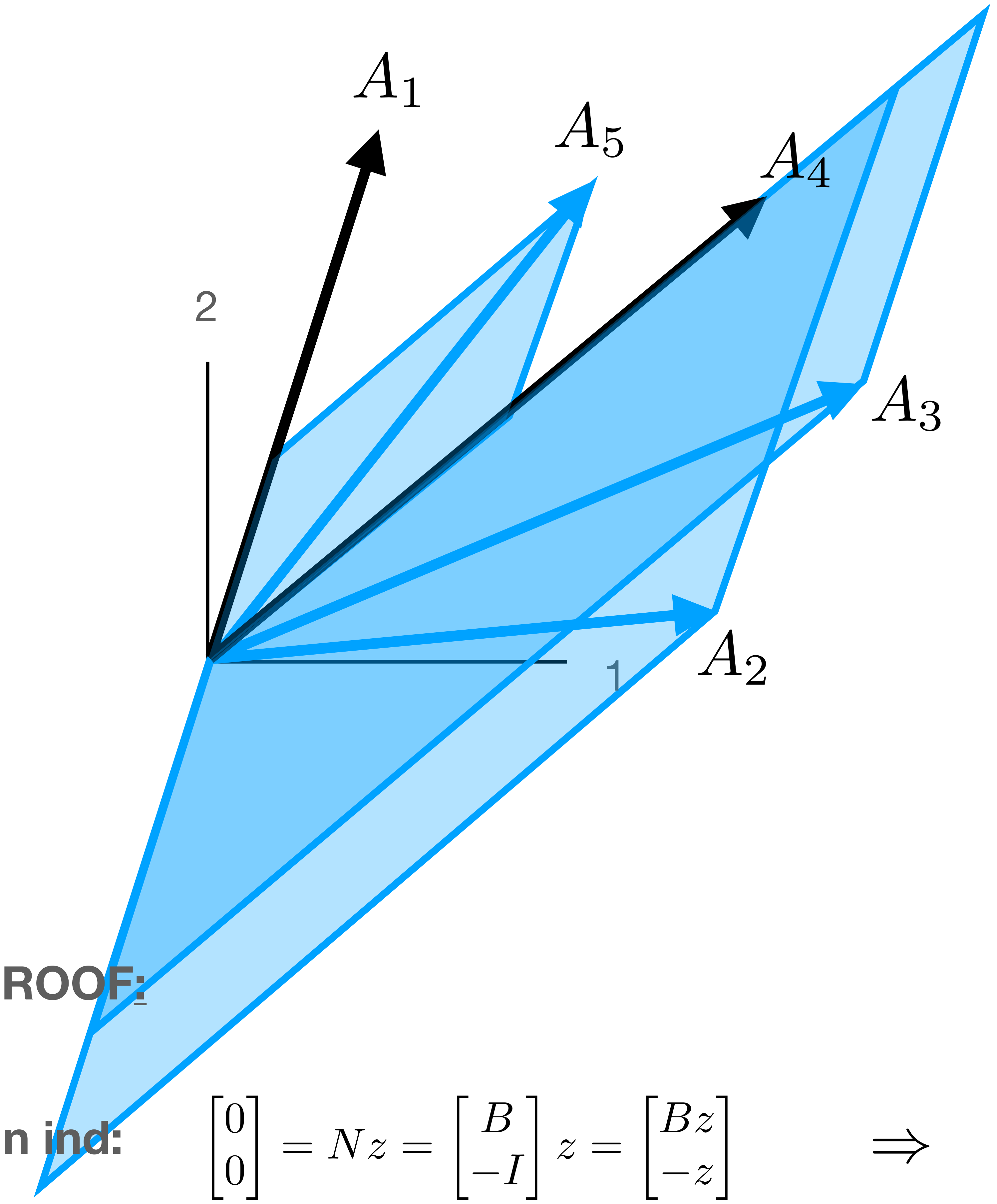
REVIEW: Nullspace - Column Geometry (Computation)

$$A = \left[\; A' \quad A'' \;\right] = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$



A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{15} \\ B_{42} & B_{43} & B_{45} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_4 B_{42} & A_1 B_{13} + A_4 B_{43} & A_1 B_{15} + A_4 B_{45} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$