Review

— Limit set of Directed AP $\dot{x} = -L(D)x$ digraph.

$\mathcal{N}(L(D)) = \mathcal{A} \to$ agreement set $= \text{span}\{\mathbb{1}\}$

$\Leftrightarrow$ D Contains a rooted outbranching as subgraph.

[Proof used matrix-Tree theorem + charac. polyn of $L(D)$]

— Convergence of DAP:

follows by the Geršgorin Disk Theorem.

— If D has a rooted out-branching subgraph

$\Rightarrow x(t) \to (q_1^T x_0) \cdot \mathbb{1}$ where $q_1^T \mathbb{1} = 1$.

In addition, if D is balanced,

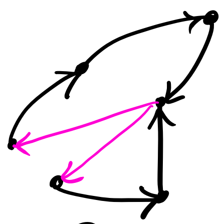
$\Rightarrow x(t) \to \left(\frac{\mathbb{1}^T x_0}{n}\right) \cdot \mathbb{1}$ (average consensus)

Question: This provides sufficient conditions for convergence of DAP to average consensus.
But, Are these conditions also <u>necessary</u>?!

In fact, something more stronger is true.

Def: A digraph is "strongly connected" if between every two verticies, there exists a directed path.

Ex:



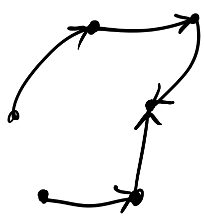
D

D (without pink edges) is not strongly connected.

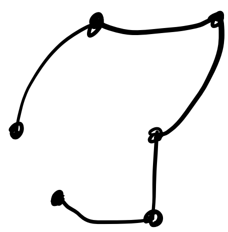
But, adding these two edges makes the resulting digraph strongly connected.

Def: A digraph is "weakly connected" if its undirected / disoriented version is connected.

Ex:



D



disoriented version of D

D is NOT strongly connected but it is weakly connected.

Fact: If D is strongly connected, then it's weakly connected.

Thm: The DAP on D reaches the average consensus from every initial condition if and only if D is weakly connected and balanced.

Proof; $\Leftarrow$) if D is weakly connected and balanced then it has to be strongly connected (why?).

Euler's Theorem

Therefore, D has a rooted out-branching subgraph.

Thus, because D is balanced, by the above corollary, DAP converges to the average consensus. ✓.

$\Rightarrow$) Conversely, suppose the convergence to average consensus is achieved by DAP, i.e.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-L(D)t} x(0) = \frac{1^T x(0)}{n} . 1 = \frac{1}{n} 1 1^T x_0 ,$$

for every $x(0) \in \mathbb{R}^n$. This implies

$$\left[\lim_{t\to\infty} e^{-L(D)t} - \frac{1}{n} 1 1^T\right] x(0) = 0 , \quad \forall x(0) \in \mathbb{R}^n.$$

Thus, $\lim_{t\to\infty} e^{-L(D)t} = \frac{1}{n} 1 1^T$. Now, note that left/right eigenvectors of $L(D)$, $e^{-L(D)t}$ and $\frac{1}{n} 1 1^T$ must match,

because: $e^{-L(D)t} = P e^{-J(D)t} P^{-1}$ where $L(D) = P J(D) P^{-1}$,

and

$$\frac{1}{n} 1 1^T = \lim_{t\to\infty} e^{-L(D)t} = P \underbrace{\left(\lim_{t\to\infty} e^{-J(D)t}\right)}_{\text{convergent}} P^{-1} .$$

Therefore, $\mathbb{1}$ has to be left and right eigenvector of $L(D)$. By definition, $L(D)\,\mathbb{1} = 0$. Assume

$$\mathbb{1}^T L(D) = \alpha \mathbb{1}^T \quad \text{for some } \alpha.$$

But then

$$0 = (L(D)\mathbb{1})^T \mathbb{1} = \mathbb{1}^T L(D)^T \mathbb{1} = \mathbb{1}^T (\mathbb{1}^T L(D))^T = \mathbb{1}^T (\alpha \mathbb{1}^T)^T = \alpha \cdot n$$

$$\Rightarrow \alpha = 0 \quad \Rightarrow \mathbb{1}^T L(D) = 0 \quad \Rightarrow D \text{ is balanced.}$$

Next, we have to show that D is weakly connected. Note:

$$e^{-L(D)t} = P\, e^{-J(D)t} P^{-1}$$

$$= \begin{bmatrix} | & | & & | \\ \frac{\mathbb{1}}{\sqrt{n}} & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{-J(0)t} & & & 0 \\ & e^{-J(\lambda_2)t} & & \\ & & \ddots & \\ 0 & & & e^{-J(\lambda_n)t} \end{bmatrix} \begin{bmatrix} - & \mathbb{1}^T/\sqrt{n} & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \longrightarrow \frac{\mathbb{1}}{\sqrt{n}} \frac{\mathbb{1}^T}{\sqrt{n}}$$

Thus, we can conclude that $J(0) = 0$, i.e., 0 has algebraic multiplicity one.

Thus, if $L(D)v = 0 \Rightarrow v \in \text{span}\{\mathbb{1}\} \Rightarrow \dim \mathcal{N}(L(D)) = 1$

$$\Rightarrow \text{rank}(L(D)) = n-1$$

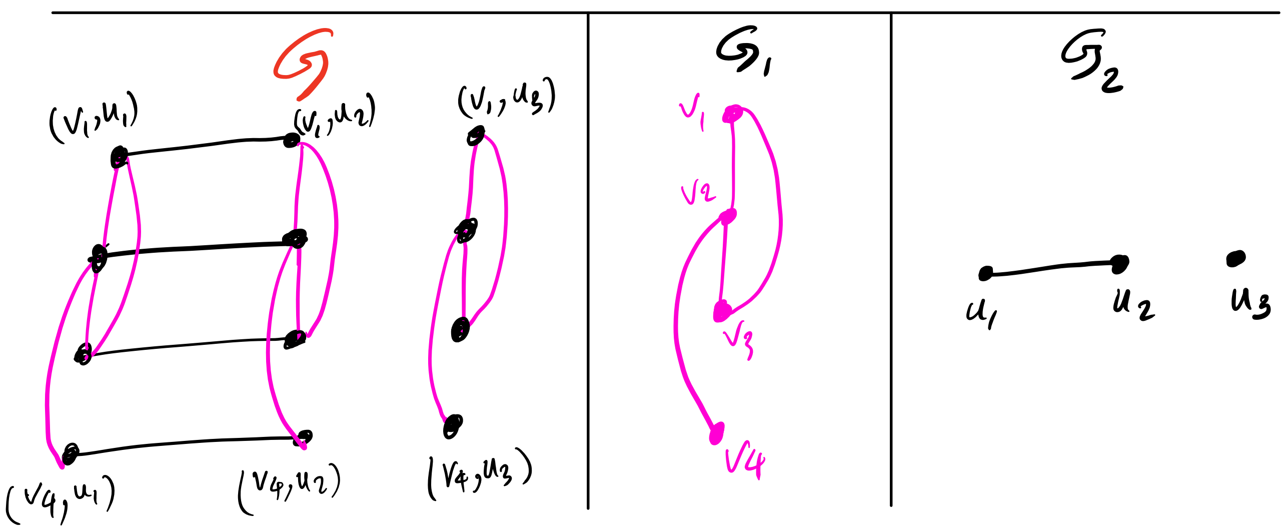
[Prop 3.8] $\Rightarrow$ D has a rooted out-branching

$\Rightarrow$ G is connected. ∎

So far, we have characterized the behaviors of both the AP on undirected graph G and the DAP on digraph D.

Next, we will focus on the (undirected) AP dynamics.

Now, suppose G has a "factorizable structure", e.g.



Q: Can we explain the AP dynamics on G as a function of the AP dyn. on G_1 and G_2?

Yes, we can. → Factorization Lemma.

Def: Given $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ we define the Cartesian Product

$$G = G_1 \square G_2$$

as a graph with vertex set $V_1 \times V_2$, where vertices (v_1, v_2) and (v_1', v_2') are adjacent $\Longleftrightarrow$

$$\begin{cases} v_1 = v_1' \text{ and } (v_2, v_2') \in E_2; \\ \text{or} \\ v_2 = v_2' \text{ and } (v_1, v_1') \in E_1; \end{cases}$$

Properties:

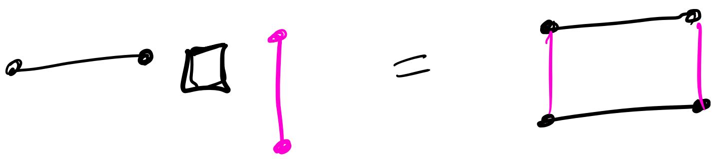
1) $G_1 \square G_2 \overset{\text{isomorphic}}{\simeq} G_2 \square G_1$

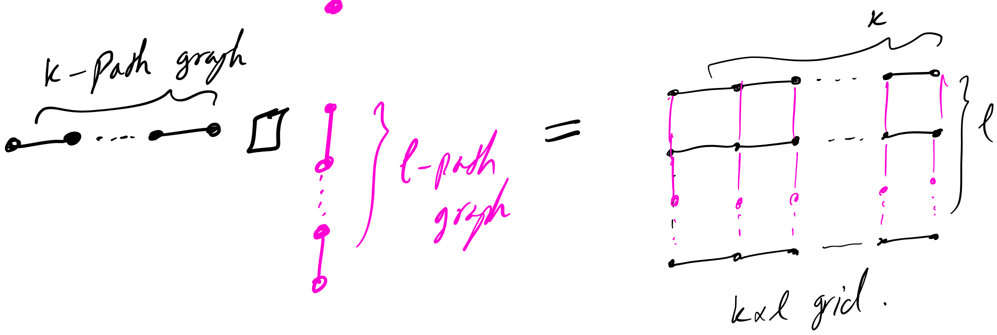
the same graph up to relabeling.

2) $(G_1 \square G_2) \square G_3 \simeq G_1 \square (G_2 \square G_3)$

3) if G_1, G_2 are both connected, then so is $G_1 \square G_2$.

e.g.

1) 

2) 

3) The motivating example $G = G_1 \square G_2$.

Prime factorization of graphs:

Recall prime factorization of Natural numbers $12 = 2\times2\times3$.

Def: we say a graph is "Prime" if it cannot be factored as a Cartesian product of non-trivial graphs.

[trivial graph is just a single node].

Exa: Tree graphs, complete graphs (why ?!)

Thm 3.24 [Mesbahi '10]

Every connected graph can be factored as a Cartesian product of prime graphs and it's unique up to reordering.

Recall the Kronecker product of two matrices

$$\underset{n\times m}{A} \otimes \underset{k\times l}{B} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & \ddots & \\ a_{n1}B & \cdots & & a_{nn}B \end{bmatrix}_{nk \times ml}$$

properties:

✓ $A\otimes(B+C) = A\otimes B + A\otimes C$

✓ $(\alpha A)\otimes B = \alpha(A\otimes B) = A\otimes(\alpha B)$

✓ $(A\otimes B)\otimes C = A\otimes(B\otimes C)$

✓ $A\otimes 0 = 0 = 0\otimes A$

✓✓ $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$

Lemma: suppose $\begin{cases} G_1 \text{ has } n \text{ vertices} \\ G_2 \text{ '' } m \text{ '' } \end{cases}$. Then,

$$L(G_1 \square G_2) = \underbrace{L(G_1)\otimes I_m + I_n \otimes L(G_2)}$$

this is called Kronecker sum denoted by

$L(G_1) \oplus L(G_2)$.