Review: $\dot{x} = f(x(t))$

- Lyapunov theorem:

a radially unbounded Lyapunov function $\Rightarrow$ global asymptotic stability.

$\dot{V} < 0$

strict ($x \neq 0$)

- For (AP) dynamics:

$$\dot{x} = -L(G)x, \quad \text{define: } V(x(t)) := \tfrac{1}{2} x^T(t)\, x(t)$$

$$\Rightarrow \dot{V}(t) = -x^T(t)\, L(G)\, x(t) \leq 0$$

not strict.

$V(t)$ is NOT a Lyapunov function;

Instead, we call it a "weak Lyapunov function".

- Thm: [LaSalle's Invariance Principle] $\leftarrow$ a generalization of Lyap. thm.

V: weak Lyapunov func s.t. $V(x) \to \infty$ as $\|x\| \to \infty$.

M: largest invariant set contained in $\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$.

Then, $\inf_{y \in M} \|x(t) - y\| \to 0$ as $t \to \infty$.

Now that we've learned a new tool, let's see how it applies to our problem!

Back to our AP dynamics

if G is connected.

$$\{x \in \mathbb{R}^n \mid \dot{V}(t)=0\} = \{x \in \mathbb{R}^n \mid x^T L(G) x = 0\} = \operatorname{span}\{\mathbf{1}\}.$$

and as $\dot{x}(t)=0$ if $x(t) \in \operatorname{span}\{\mathbf{1}\} \Rightarrow M = \operatorname{span}\{\mathbf{1}\}$

Thus, by LaSalle's Invariance Principle,

$$x(t) \longrightarrow \operatorname{span}\{\mathbf{1}\}.$$

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what about the DAP dynamics?

$\dot{x}(t) = -L(D)\, x(t)$ define $V(x(t)) = \frac{1}{2} x^T(t)\, x(t)$

not symm.

$$\Rightarrow \dot{V}(t) = x^T(t)\, \dot{x}(t) = -x^T(t)\, L_{in}(D)\, x(t)$$

By Geršgorian disk thm ≤ 0

not strictly < 0. $\Rightarrow$ weak Lyapunov funct.

If D is strongly connected then, the largest invariant set in:

$$\{x \in \mathbb{R}^n \mid \dot{V}(t)=0\} = \{x \mid x^T (L(D) + L(D)^T) x = 0\}$$

is the null space of $L(D)$ which is $\operatorname{span}\{\mathbf{1}\}$. (why?)

$\Rightarrow$ By, LaSalle's Inv. Prin., $x(t) \longrightarrow \operatorname{span}\{\mathbf{1}\}$.

what if D is not strongly connected, yet contains a rooted out-branching?

Redefine a (discrete) weak Lyapunov function as

$$V(z(k)) = \max_{i \in |G|} (z_i(k)) - \min_{i \in |G|} (z_i(k))$$

where, $z(k) = x(\delta k)$ for some $\delta > 0$.

→ see ch. 4.1.2 in [mesbahi'10].

Switched Agreement Protocol:

Consider finitely many strongly connected digraphs

$$\{D_1, \ldots, D_k\}.$$

switched AP

Suppose $\dot{x}(t) = -L(D_i)\, x(t)$ with $i \in \{1, .., k\}$.

This is a "switched linear system" and described by

"Differential Inclusion" $\dot{x}(t) \in \{-L(D_i)\, x(t) \mid i \in \{1, .., k\}\}$.

Considering $V(x(t)) = \frac{1}{2} x^T(t)\, x(t)$, we get

$$\dot{V}(t) \in \{-x^T(t)\, L(D_i)\, x(t) \mid i \in \{1, \ldots, k\}\}.$$

where each dynamic vanishes on:

$$F_i = \{ x \in \mathbb{R}^n \mid x^T (L(D_i) + L(D_i)^T) x \}.$$

But, as each D_i is strongly connected,

$$F_i = \text{span}\{\mathbf{1}\} \quad \text{for every } i \in \{1, \dots, k\}$$

we call $V(t)$ here a "common weak Lyapunov function" for the switched agreement protocol.

$\Rightarrow$ A generalization of LaSalle's Inv. principle [Thm A.9 in Meshahi '10] still implies that $x(t) \longrightarrow \text{span}\{\mathbf{1}\}$.

Thm A.9: Suppose V is a common weak Lyapunov function for the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \sigma(t) \in S = \{1, \dots, k\}$$

switching mechanism.

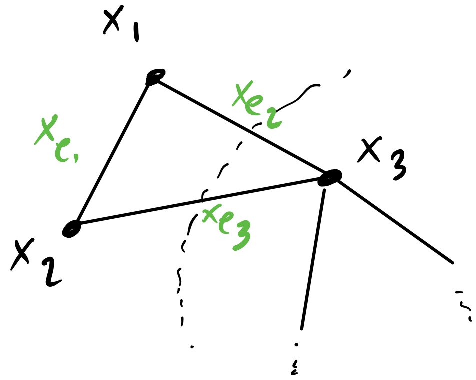
a generalization of LaSalle's Inv. princ.

Let M_i be the largest invariant set under mode i that is contained in

$$\left\{ x \in \mathbb{R}^n \,\middle|\, \left[\frac{\partial V(x)}{\partial x} \right]^T f_i(x) = 0 \right\}.$$

If $M_i = M_j = M^*$ for all $i, j \in S$, then $x(t) \longrightarrow M^*$ as $t \to \infty$.

Edge Agreement (Consensus):



Assign a state to each edge:

$$x_e(t) = \begin{bmatrix} x_{e_1}(t) \\ \vdots \\ x_{e_m}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t) \\ x_1(t) - x_3(t) \\ \vdots \end{bmatrix} = D(G)^T x(t)$$

\# of edges.

an arbitrary choice of orientation.

Now,

$$\dot{x}_e(t) = D(G)^T \dot{x}(t) = -D(G)^T L(G)\, x(t)$$
$$= -D(G)^T D(G)\, D(G)^T x(t) = -L_e(G)\, x_e(t).$$

$$\boxed{\text{Edge agreement protocol on } G: \quad \dot{x}_e(t) = -L_e(G)\, x_e(t)}$$

- If G is connected, then

$$x_e = 0 \iff x \in \text{span}\{\mathbb{1}\}$$

- In this case, we have shown that $x(t) \to \text{span}\{\mathbb{1}\}$ which implies that $x_e(t) \to 0$.

Role of cycles in Edge AP :

Recall the decomposition of the incidence matrix (up to edge relabeling)

$$D(G) = [\ \underbrace{D(G_T)}_{\text{incidence matrix of the underlying spanning tree}} \quad \underbrace{D(G_C)}_{\text{the remaining edges creating cycles.}}\ ]$$

now, also decompose $x_e(t)$ accordingly to $x_e(t) = \begin{bmatrix} x_T(t) \\ x_C(t) \end{bmatrix}$.

Then,

$$L_e(G) = D^T(G)\, D(G) = \begin{bmatrix} \overset{\text{edge Laplacian of spanning tree}}{L_e(G_T)} & D(G_T)^T D(G_C) \\ D(G_C)^T D(G_T) & \underset{\text{edge Lap. of the the remaining graph.}}{L_e(G_C)} \end{bmatrix}$$

and thus the edge dynamics decomposes as :

$$\begin{cases} \dot{x}_T(t) = -L_e(G_T)\, x_T(t) - D(G_T)^T D(G_C)\, x_C(t) \\ \dot{x}_C(t) = -L_e(G_C)\, x_C(t) - D(G_C)^T D(G_T)\, x_T(t) \end{cases}$$

But we know that $x_T \in \mathbb{R}^{|G|-1}$ and any edge in the cycle can be constructed using the ones in the spanning tree. So,

"Is there a reduced model for edge AP ?!".

Recall that

$$D(G) = [D(G_T) \quad D(G_C)] = D(G_T)[I \quad M] =: D(G_T) R.$$

where $D(G_C) = D(G_T) M$ ← each column of M says how to traverse the spanning tree to construct an edge in G_C.

i.e., by construction, note that $\boxed{x_C^T(t) = x_T^T(t) M}$. (why?)

Therefore:

$$\dot{x}_T(t) = -L_e(G_T) x_T(t) - D(G_T)^T D(G_C) x_C(t)$$

$$= -L_e(G_T) x_T(t) - D(G_T)^T D(G_T) M x_C(t)$$

$$= -L_e(G_T) [x_T(t) + M x_C(t)]$$

$$= -L_e(G_T) [x_T(t) + M M^T x_T(t)]$$

$$= -L_e(G_T) [I + M M^T] x_T(t)$$

$$= -L_e(G_T) R R^T x_T(t)$$

So, Edge AP follows:

the independent spanning tree dyn. $\boxed{\dot{x}_T(t) = -L_e(G_T) R R^T x_T(t)}$

the lin. depen. cycle dyn. $\boxed{x_C(t) = M^T x_T(t)}$

Lastly, we can interpret Edge AP as a feedback system between the spanning tree edges and cycle edges.

$$\dot{x}_T(t) = -L_e(G_T)\,x_T(t) - D(G_T)^T D(G_c)\,\underbrace{x_c(t)}_{= M^T x_T(t)}$$