Shift, Circulant, & DFT Matrices

Linear Algebra

Major Contributions: Anant Sahai,
Preetum Nakkiran

Winter 2022 - Dan Calderone

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

for $k = 0, 1, \ldots, T-1$

$S \in \mathbb{R}^{T \times T}$ $\quad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix}$ $\quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$ …step by t time step

Eigenvalue k Eigenvector k (frequency k)

…step by 1 time step

$$x^+ = Sx$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

$$e^{i2\pi\frac{k}{T}} \qquad \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

t-th element $\leftarrow e^{i2\pi\frac{tk}{T}}$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}},\ e^{i2\pi\frac{1}{T}},\ e^{i2\pi\frac{2}{T}},\ \ldots\ e^{i2\pi\frac{T-2}{T}},\ e^{i2\pi\frac{T-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(T-2)}{T}} & e^{i2\pi\frac{1(T-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(T-2)}{T}} & e^{i2\pi\frac{2(T-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(T-2)}{T}} & e^{i2\pi\frac{(T-2)(T-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(T-2)}{T}} & e^{i2\pi\frac{(T-1)(T-1)}{T}} \end{bmatrix}$$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Eigenvectors:

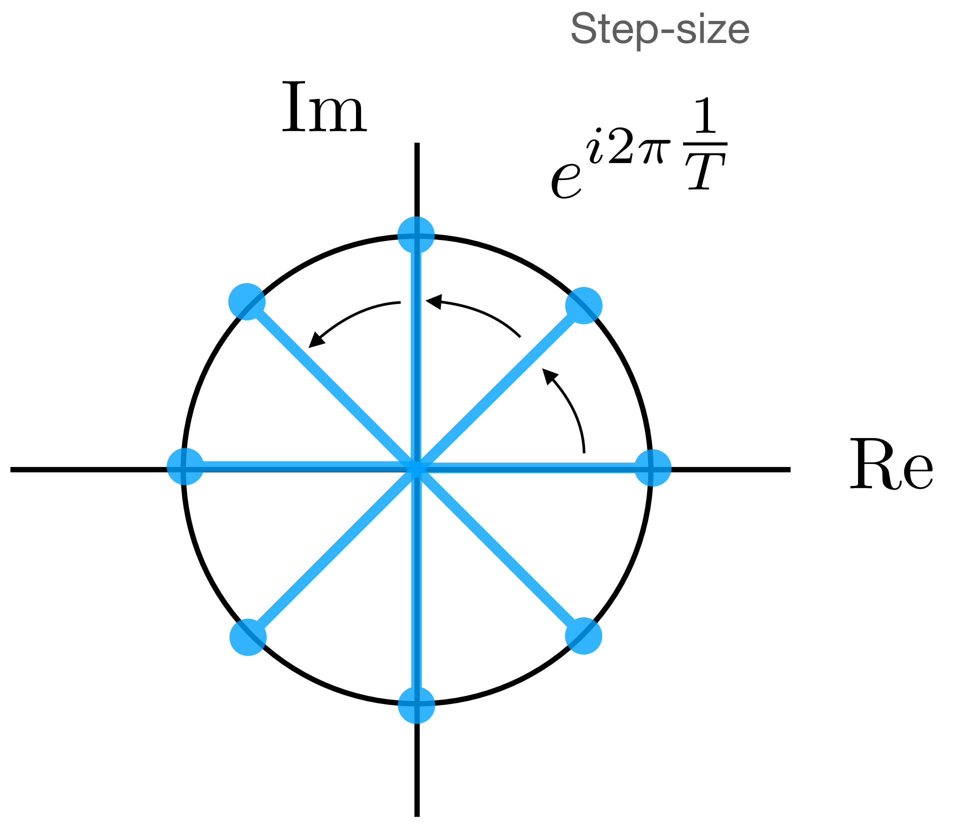
$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}},\ e^{i2\pi\frac{1}{T}},\ e^{i2\pi\frac{2}{T}},\ \ldots\ e^{i2\pi\frac{T-2}{T}},\ e^{i2\pi\frac{T-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(T-2)}{T}} & e^{i2\pi\frac{1(T-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(T-2)}{T}} & e^{i2\pi\frac{2(T-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(T-2)}{T}} & e^{i2\pi\frac{(T-2)(T-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(T-2)}{T}} & e^{i2\pi\frac{(T-1)(T-1)}{T}} \end{bmatrix}$$



1-rotation

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, e^{i2\pi\frac{1}{T}}, e^{i2\pi\frac{2}{T}}, \ldots \quad e^{i2\pi\frac{T-2}{T}}, \quad e^{i2\pi\frac{T-1}{T}},$

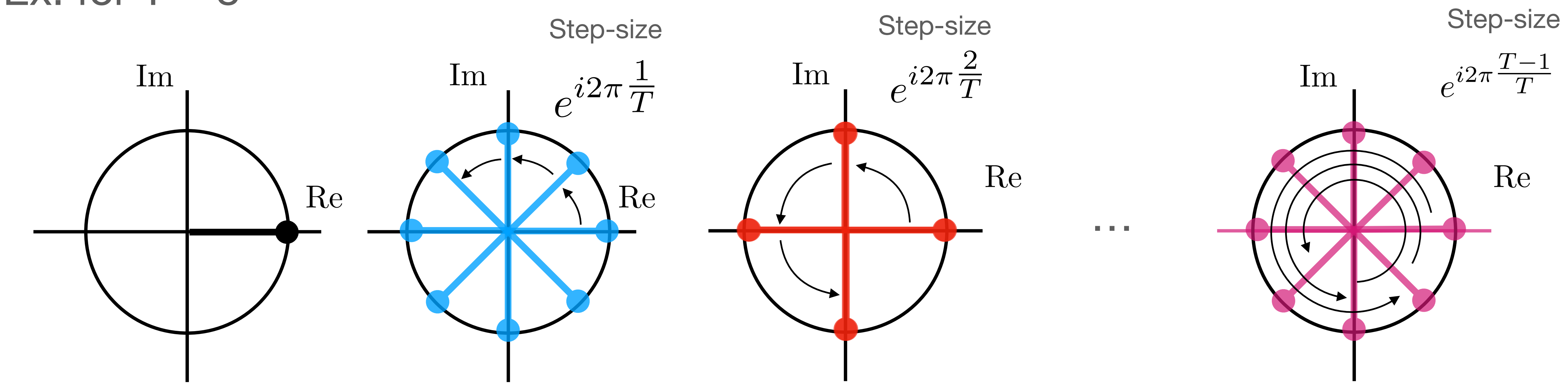
Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(T-2)}{T}} & e^{i2\pi\frac{1(T-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(T-2)}{T}} & e^{i2\pi\frac{2(T-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(T-2)}{T}} & e^{i2\pi\frac{(T-2)(T-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(T-2)}{T}} & e^{i2\pi\frac{(T-1)(T-1)}{T}} \end{bmatrix}$$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

Ex: for T = 8



0-rotations 1-rotation 2-rotations (T-1)-rotations

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

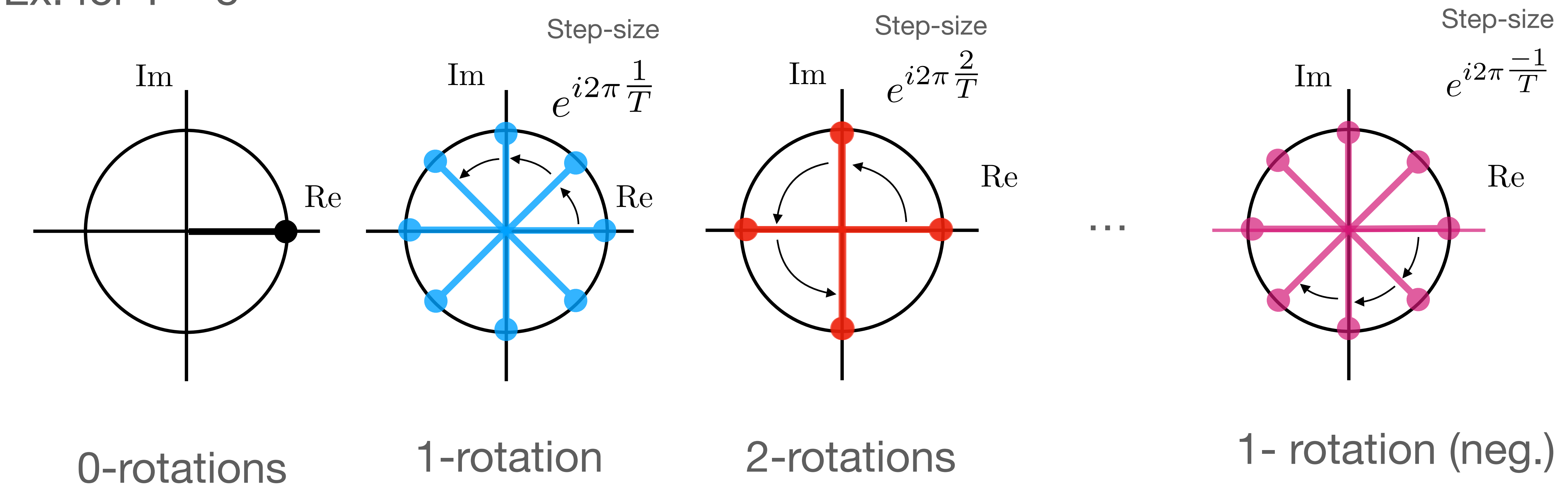
Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}},\ e^{i2\pi\frac{1}{T}},\ e^{i2\pi\frac{2}{T}},\ \ldots\ e^{i2\pi\frac{-2}{T}},\ e^{i2\pi\frac{-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Ex: for T = 8



Shift Matrix

… for a discrete time periodic signal

… represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \dots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}},\ e^{i2\pi\frac{1}{T}},\ e^{i2\pi\frac{-1}{T}},\ e^{i2\pi\frac{2}{T}},\ e^{i2\pi\frac{-2}{T}},\ \dots$

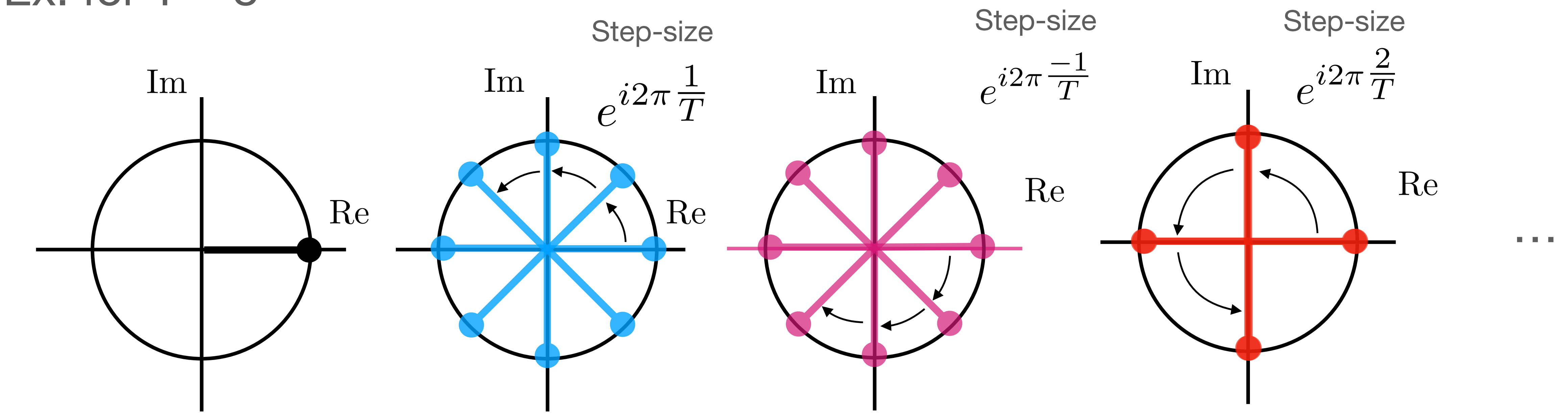
Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(-1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & e^{i2\pi\frac{1(-2)}{T}} & \cdots \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(-1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & e^{i2\pi\frac{2(-2)}{T}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & e^{i2\pi\frac{(T-2)(-2)}{T}} & \cdots \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & e^{i2\pi\frac{(T-1)(-2)}{T}} & \cdots \end{bmatrix}$$

Eigenvectors:

$$\begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix} e^{i2\pi\frac{k}{T}} = \begin{bmatrix} e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ e^{i2\pi\frac{3k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-1)k}{T}} \\ e^{i2\pi\frac{0k}{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{i2\pi\frac{0k}{T}} \\ e^{i2\pi\frac{1k}{T}} \\ e^{i2\pi\frac{2k}{T}} \\ \vdots \\ e^{i2\pi\frac{(T-2)k}{T}} \\ e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}$$

Ex: for T = 8



0-rotations

1-rotation

1- rotation (neg.)

2-rotations

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, e^{i2\pi\frac{1}{T}}, e^{i2\pi\frac{2}{T}}, \ldots \quad e^{i2\pi\frac{-2}{T}}, e^{i2\pi\frac{-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$S = U\mathrm{dg}(U_1)U^* \qquad U^*U = UU^* = I$$

$$S = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi\frac{1}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi\frac{2}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi\frac{(T-2)}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi\frac{(T-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, e^{i2\pi\frac{1}{T}}, e^{i2\pi\frac{2}{T}}, \ldots \; e^{i2\pi\frac{-2}{T}}, e^{i2\pi\frac{-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$S = U\mathrm{dg}(U_1)U^* \qquad U^*U = UU^* = I$$

$$S = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi\frac{1}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi\frac{2}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi\frac{(T-2)}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi\frac{(T-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Shift Matrix

… for a discrete time periodic signal … represents a step forward in time

$$S \in \mathbb{R}^{T\times T} \qquad S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \qquad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$$

…step by t time step

$$x^+ = Sx$$

…step by 1 time step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

Frequency: $k = 0, 1, \dots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, \; e^{i2\pi\frac{1}{T}}, \; e^{i2\pi\frac{2}{T}}, \; \dots \; e^{i2\pi\frac{-2}{T}}, \; e^{i2\pi\frac{-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$S^t = U\mathrm{dg}(U_1)^t U^* = U\mathrm{dg}(U_t)U^* \qquad U^*U = UU^* = I$$

$$S^t = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{i2\pi\frac{1t}{T}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{i2\pi\frac{2t}{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i2\pi\frac{(T-2)t}{T}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e^{i2\pi\frac{(T-1)t}{T}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

… all shifts of a periodic signal…

$c \in \mathbb{R}^T$ periodic signal

$C \in \mathbb{R}^{T\times T}$ circulant matrix

$S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$ …step by t time step

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix} \qquad C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts: $C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $e^{i2\pi\frac{0}{T}}, e^{i2\pi\frac{1}{T}}, e^{i2\pi\frac{2}{T}}, \ldots \; e^{i2\pi\frac{-2}{T}}, e^{i2\pi\frac{-1}{T}},$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Discrete Convolution: $y = (c * x) = Cx$

Impulse Response: $c = \begin{bmatrix} c_0 & c_1 & \cdots & c_{T-1} \end{bmatrix}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{T-2} \\ y_{T-1} \end{bmatrix} = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

$$y_t = \sum_\tau c_{t-\tau} x_\tau$$

Note: Impulse response at time t… selects that column…

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix} = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Circulant Matrix

… all shifts of a periodic signal…

$c \in \mathbb{R}^T$ $\quad C \in \mathbb{R}^{T\times T}$ $\quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$ …step by t time step

periodic signal

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix}$$

circulant matrix

$$C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts: $C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $\sqrt{T}\,U_0^* c \quad \sqrt{T}\,U_1^* c \quad \sqrt{T}\,U_2^* c \quad \ldots \quad \sqrt{T}\,U_{T-2}^* c \quad \sqrt{T}\,U_{T-1}^* c$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$C = \sum_{t=0}^{T-1} c_t S^t = U \left(\sum_{t=0}^{T-1} c_t \mathrm{dg}(U_t) \right) U^* = \sqrt{T}\, U \mathrm{dg}(U^* c) U^* \qquad U^*U = UU^* = I$$

$$C = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \sqrt{T} \begin{bmatrix} U_0^* c & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_1^* c & 0 & \cdots & 0 & 0 \\ 0 & 0 & U_2^* c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{T-2}^* c & 0 \\ 0 & 0 & 0 & \cdots & 0 & U_{T-1}^* c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

… all shifts of a periodic signal…

$c \in \mathbb{R}^T$ periodic signal

$C \in \mathbb{R}^{T\times T}$ circulant matrix

$S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$ …step by t time step

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix} \qquad C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts: $C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2}S^{T-2} + c_{T-1}S^{T-1}$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $\sqrt{T}\,U_0^* c$ $\quad \sqrt{T}\,U_1^* c$ $\quad \sqrt{T}\,U_2^* c$ $\quad \ldots \quad \sqrt{T}\,U_{T-2}^* c$ $\quad \sqrt{T}\,U_{T-1}^* c$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

$$y = (c * x) = Cx = \sqrt{T}\,U \mathrm{dg}(U^*c)U^*x \qquad \sqrt{T}\,U^*y = \mathrm{dg}\left(\sqrt{T}\,U^*c\right)\sqrt{T}\,U^*x$$

$$C = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \sqrt{T} \begin{bmatrix} U_0^*c & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_1^*c & 0 & \cdots & 0 & 0 \\ 0 & 0 & U_2^*c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{T-2}^*c & 0 \\ 0 & 0 & 0 & \cdots & 0 & U_{T-1}^*c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \frac{1}{\sqrt{T}}$$

Circulant Matrix

… all shifts of a periodic signal…

$c \in \mathbb{R}^T$ $\quad C \in \mathbb{R}^{T \times T}$ $\quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$ …step by t time step

periodic signal $\quad$ circulant matrix

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{T-2} \\ c_{T-1} \end{bmatrix} \qquad C = \begin{bmatrix} c_0 & c_{T-1} & c_{T-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{T-1} & \cdots & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{T-2} & c_{T-3} & c_{T-4} & \cdots & c_0 & c_{T-1} \\ c_{T-1} & c_{T-2} & c_{T-3} & \cdots & c_1 & c_0 \end{bmatrix}$$

Sum of shifts: $C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{T-2} S^{T-2} + c_{T-1} S^{T-1}$

Frequency: $k = 0, 1, \ldots, T-1$

Eigenvalues: $\sqrt{T}\, U_0^* c \quad \sqrt{T}\, U_1^* c \quad \sqrt{T}\, U_2^* c \quad \ldots \quad \sqrt{T}\, U_{T-2}^* c \quad \sqrt{T}\, U_{T-1}^* c$

Eigenvectors:

$$U = \frac{1}{\sqrt{T}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{i2\pi\frac{1(1)}{T}} & e^{i2\pi\frac{1(2)}{T}} & \cdots & e^{i2\pi\frac{1(-2)}{T}} & e^{i2\pi\frac{1(-1)}{T}} \\ 1 & e^{i2\pi\frac{2(1)}{T}} & e^{i2\pi\frac{2(2)}{T}} & \cdots & e^{i2\pi\frac{2(-2)}{T}} & e^{i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{i2\pi\frac{(T-2)1}{T}} & e^{i2\pi\frac{(T-2)2}{T}} & \cdots & e^{i2\pi\frac{(T-2)(-2)}{T}} & e^{i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{i2\pi\frac{(T-1)1}{T}} & e^{i2\pi\frac{(T-1)2}{T}} & \cdots & e^{i2\pi\frac{(T-1)(-2)}{T}} & e^{i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix}$$

Diagonalization:

Convolution in time domain…

$$y = (c * x) = Cx = \sqrt{T}\, U \mathrm{dg}(U^* c) U^* x$$

Multiplication in frequency domain…

$$\underbrace{\sqrt{T}\, U^* y}_{\hat{y}} = \mathrm{dg}\big(\underbrace{\sqrt{T}\, U^* c}_{\hat{c}}\big) \underbrace{\sqrt{T}\, U^* x}_{\hat{x}}$$

elementwise…

$$\hat{y}_k = \hat{c}_k \hat{x}_k$$

Discrete Fourier Transform

$$\hat{x} = \sqrt{T}\, U^* x = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e^{-i2\pi\frac{1(1)}{T}} & e^{-i2\pi\frac{1(2)}{T}} & \cdots & e^{-i2\pi\frac{1(-2)}{T}} & e^{-i2\pi\frac{1(-1)}{T}} \\ 1 & e^{-i2\pi\frac{2(1)}{T}} & e^{-i2\pi\frac{2(2)}{T}} & \cdots & e^{-i2\pi\frac{2(-2)}{T}} & e^{-i2\pi\frac{2(-1)}{T}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & e^{-i2\pi\frac{(T-2)1}{T}} & e^{-i2\pi\frac{(T-2)2}{T}} & \cdots & e^{-i2\pi\frac{(T-2)(-2)}{T}} & e^{-i2\pi\frac{(T-2)(-1)}{T}} \\ 1 & e^{-i2\pi\frac{(T-1)1}{T}} & e^{-i2\pi\frac{(T-1)2}{T}} & \cdots & e^{-i2\pi\frac{(T-1)(-2)}{T}} & e^{-i2\pi\frac{(T-1)(-1)}{T}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \end{bmatrix}$$

$$\hat{x}_k = \sum_{t=0}^{T-1} x_t e^{-i2\pi\frac{tk}{T}}$$

Circulant Matrix - parallels with continuous time

	Discrete Time	Continuous Time
Signals	$c, x, y \in \mathbb{R}^T$	$c(t), x(t), y(t) \quad t \in [0, \Delta t T]$
Eigen functions	$U_k = \begin{bmatrix} e^{i2\pi\frac{0k}{T}} & e^{i2\pi\frac{1k}{T}} & \dots & e^{i2\pi\frac{(T-1)k}{T}} \end{bmatrix}^T \in \mathbb{C}^n$	$e^{i2\pi\omega t}$
Time evolution	$S = \begin{bmatrix} \mathbf{0} & I_{T-1} \\ 1 & \mathbf{0} \end{bmatrix} \quad S^t = \begin{bmatrix} \mathbf{0} & I_{T-t} \\ I_t & \mathbf{0} \end{bmatrix}$	$\frac{d}{dt}x = \dot{x}$ …infinitesimal; $\int_0^{\Delta t} e^{\Delta t - \tau}(\cdot)\ d\tau$ …integration over a time step; $\int_0^{t\Delta t} e^{t\Delta t - \tau}(\cdot)\ d\tau$ …integration over t time steps
Laplace/ Fourier Transform	$\hat{x} = \sqrt{T}\, U^* x \qquad \hat{x}_k = \sqrt{T} \sum_{t=0}^{T-1} x_t e^{-i2\pi\frac{tk}{T}}$	**Laplace** $\hat{x}(s) = \mathcal{L}(x(t)) = \int_0^{\infty} e^{-st} x(t)\ dt$; **Fourier** $\hat{x}(\omega) = \mathcal{F}(x(t)) = \int_0^{\infty} e^{-i2\pi\omega t} x(t)\ dt$
Inverse Transforms	$x = \frac{1}{\sqrt{T}} U \hat{x} \qquad x_t = \frac{1}{\sqrt{T}} \sum_{k=0}^{T-1} \hat{x}_k e^{i2\pi\frac{tk}{T}}$	**Inv. Fourier** $x(t) = \mathcal{F}^{-1}(\hat{x}(\omega)) = \int_{-\infty}^{\infty} e^{i2\pi\omega t}\hat{x}(\omega)\ d\omega$ (Laplace, Fourier, Inv. Fourier: function inner product)
Convolution	$y = (c * x) = Cx \qquad y_t = \sum_{\tau} c_{t-\tau} x_\tau$	$y(t) = \int_0^t c(t-\tau)x(\tau)\ d\tau$
	$y = (c * x) = \sqrt{T}\, U \mathrm{dg}(U^* c) U^* x$	$y(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{c\}(s) \cdot \mathcal{L}\{x\}(s)\right\}$
	$\hat{y} = \mathrm{dg}(\hat{c})\hat{x} \qquad \hat{y}_k = \hat{c}_k \hat{x}_k$	$\hat{y}(s) = \hat{c}(s)\hat{x}(s)$
Plancherel/ Parseval's Theorem	$\langle \hat{y}, \hat{x} \rangle = \hat{y}^* \hat{x} = \frac{\sqrt{T}}{\sqrt{T}} y^* U U^* x = y^* x$	$\langle \hat{y}, \hat{x} \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(i\omega)^* \hat{x}(i\omega)\ d\omega \quad = \int_0^{\infty} y(t)^* x(t)\ dt = \langle y, x \rangle$