Review: Agreement protocol

★ (undirected) $\dot{x} = -L(G)x$ (AP)

(Limit set) ∘ $\mathcal{N}(L(G)) = \mathcal{A}$ iff G is connected.

(spectrum) ∘ $L(G) = D(G)D(G)^T$ is P.S.D.

(Convergence) ∘ $x(t) \to \left(\frac{\mathbf{1}^T x_0}{n}\right)\mathbf{1}$ as $t \to \infty$ if and only if G has a spanning tree

∘ the rate of convergence is dictated by λ_2.

∘ $\vartheta(t) = \mathbf{1}^T x(t)$ is a constant of motion.

★ (directed) $\dot{x} = -L(D)x$ (DAP)

∘ Def: rooted out-branching (no directed cycle, and has a root v_r)

(Limit set) ∘ the digraph D has a rooted out-branching as a subgraph if and only if $\mathcal{N}(L(D)) = \mathcal{A}$.

(spectrum) ∘ spectrum of $L(D)$ lies in (by Geršgorian Disk thm)

$$\{z \in \mathbb{C} \mid |z - \bar{d}_{in}(D)| \le \bar{d}_{in}(D)\}$$

max in-degree in D.

(Convergence) ∘ If D has a rooted out-branching subgraph, then

$x(t) \to (p_1 q_1^T)\, x_0$ with

p_1, q_1 being the right and left eigenvectors of $L(D)$, respectively, associated with 0 eigenvalue and $p_1^T q_1 = 1$. therefore, (as $p_1 \in \text{span}\{\mathbf{1}\}$)

compare with AP

$x(t) \to (q_1^T x_0)\,\mathbf{1}$. where $\mathbf{1}^T q_1 = 1$.

Proposition (Prop. 3.8 [Meshahi '10])

A digraph $\mathcal{D}$ on n verticies containes a rooted out-branching as a subgraph iff $\operatorname{rank}(L(\mathcal{D})) = n-1$.

In this case, $\mathcal{N}(L(\mathcal{D})) = \mathcal{A}$.

Proof: It suffices to show that "0" has algebraic multiplicity one iff $\mathcal{D}$ contains a rooted out-branching as a subgraph.

why?

Because we know that $\mathbb{1} \in \mathcal{N}(L(\mathcal{D}))$ therefore $\operatorname{rank}(L(\mathcal{D})) \leq n-1$, with equality iff 0 is a simple eigenvalue

Now, characteristic polynomial of $L(\mathcal{D})$:

$$P(\lambda) = \det(\lambda I - L(\mathcal{D}))$$
$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda_1 + a_0$$

where a_{n-k} = sum of all principal minors of $L(\mathcal{D})$ of size k.

- But $a_0 = \det(L(\mathcal{D})) = 0$.
- Thus, $\operatorname{rank} L(\mathcal{D}) = n-1$ iff $a_1 \neq 0$.
- But, $a_1 = \sum_{v \in \mathcal{D}} \det L_v(\mathcal{D})$ where $L_v(\mathcal{D})$ is a principal submatrix of $L(\mathcal{D})$ with removing the row and the column corresponding to node v.

therefore we need to understand $\det L_v(D)$.

Thm [matrix-Tree theorem (undirected graph G)]:

$$\det L_v(G) = \text{number of spanning trees in } G.$$

Thm [matrix-Tree theorem (digraph D)]:

$$\det L_v(D) = \sum_{T \in \mathcal{T}_v} \prod_{e \in T} w(e)$$

the set of spanning v out-branching subgraphs

Back to the proof:

So, $\det L_v(D) \neq 0$ iff $\exists$ a v-rooted outbranching subgraph of D.

Thus, $a_1 = \sum_v \det L_v(D) \neq 0$ iff $\exists$ a rooted outbranching subgraph of D. ▯

Thus, $A = \text{span}\{1\} \subseteq N(L(D))$

$\not\supseteq$ not true in general but it is true if D has a rooted out-branching subgraph.

Constant of motion for DAP:

$\vartheta(t) \triangleq q_1^T x(t) \Rightarrow \dot{\vartheta}(t) = q_1^T \dot{x}(t) = -q_1^T L(D)\, x(t) = 0$,

$\uparrow$ left eigenvector associated with 0.

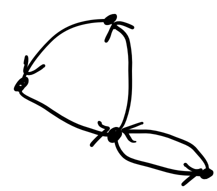
Question: when does DAP converges to the average?

we want $q_1 = \mathbb{1}$, when does this happen?!

Def: we say a digraph D is "balanced" if, for every vertex,

in-degree = out-degree.

eg:



[Note that every node might have a different degree].

Now, consider $L(D)$ for a balanced graph:

$$
L(D) = \begin{array}{c}
\\ \\ \\ \\ \text{i-th row} \rightarrow \\ \\ \\ \\
\end{array}
\begin{array}{c}
\text{i-th column} \\
\left[\begin{array}{ccccccc}
 & & & -w_{1i} & & & \\
 & & & -w_{2i} & & & \\
 & & & \vdots & & & \\
 & & & -w_{(i-1)i} & & & \\
-w_{i1} & \cdots & -w_{i(i-1)} & d_{in}(v_i) & -w_{i(i+1)} & \cdots & -w_{in} \\
 & & & -w_{(i+1)i} & & & \\
 & & & \vdots & & & \\
 & & & -w_{ni} & & &
\end{array}\right]
\end{array}
$$

recall: $v_j \xrightarrow{w_{ij}} v_i$

but $\sum_{j=1}^{n} w_{ij} = d_{in}(v_i) = d_{out}(v_i) = \sum_{j=1}^{n} w_{ji}$, Therefore

$$\mathbb{1}^T L(D) = 0 \iff q_1 \in \operatorname{span}\{\mathbb{1}\}$$

Corollary: If D contains a rooted out-branching and is balanced, then DAP reaches average consensus,

i.e. $\lim_{t \to \infty} x(t) = \frac{\mathbb{1}^T x_0}{n} \cdot \mathbb{1}$.

Proof: Recall that by hypothesis $x(t) \to (q_1^T x_0) \cdot \mathbb{1}$

with $q_1^T \mathbb{1} = 1$. As D is balanced, $q_1 \in \text{span}\{\mathbb{1}\}$

$\Rightarrow q_1 = \frac{1}{n}$. $\blacksquare$

In fact, something more stronger is true.

Def: A digraph is "strongly connected" if between every two verticies, there exists a directed path.

Def: A digraph is "weakly connected" if its undirected / disoriented version is connected.

Thm: The DAP on D reaches the average consensus from every initial condition if and only if D is weakly connected and balanced.

Proof: $\Leftarrow$) if D is weakly connected and balanced

Homework then it has to be strongly connected (why?).

Therefore, D has a rooted out-branching subgraph.

Thus, because D is balanced, by the above corollary,

DAP converges to the average consensus. ✓.

$\Rightarrow$) Conversely, suppose the convergence to average consensus is achieved by DAP, i.e.

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-L(D)t} x(0) = \frac{1^T x(0)}{n} \cdot 1 = \frac{1}{n} 1 1^T x_0,$$

for every $x(0) \in \mathbb{R}^n$. This implies

$$\left[\lim_{t\to\infty} e^{-L(D)t} - \frac{1}{n} 1 1^T\right] x(0) = 0, \quad \forall x(0) \in \mathbb{R}^n.$$

Thus, $\lim_{t\to\infty} e^{-L(D)t} = \frac{1}{n} 1 1^T$. Now, note that

left/right eigenvectors of $L(D)$, $e^{-L(D)t}$ and $\frac{1}{n} 1 1^T$ must match,

because: $e^{-L(D)t} = P\, e^{-J(D)t} P^{-1}$ where $L(D) = P J(D) P^{-1}$,

and

$$\frac{1}{n} 1 1^T = \lim_{t\to\infty} e^{-L(D)t} = P \underbrace{\left(\lim_{t\to\infty} e^{-J(D)t}\right)}_{\text{convergent}} P^{-1}.$$

Therefore, $\mathbb{1}$ has to be <u>left and right</u> eigenvector of $L(D)$. By definition, $L(D)\mathbb{1}=0$. Assume

$$\mathbb{1}^T L(D) = \alpha \mathbb{1}^T \text{ for some } \alpha.$$

But then

$$0 = (L(D)\mathbb{1})^T\mathbb{1} = \mathbb{1}^T L(D)^T \mathbb{1} = \mathbb{1}^T(\mathbb{1}^T L(D))^T = \mathbb{1}^T(\alpha \mathbb{1}^T)^T = \alpha \cdot n$$

$$\Rightarrow \alpha = 0 \Rightarrow \mathbb{1}^T L(D) = 0 \Rightarrow \underline{D \text{ is balanced.}}$$

Next, we have to show that D is weakly connected. Note:

$$e^{-L(D)t} = P e^{-J(D)t} P^{-1}$$

$$= \begin{bmatrix} | & | & & | \\ \frac{\mathbb{1}}{\sqrt{n}} & p_2 & \cdots & p_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} e^{-J(0)t} & & 0 \\ & e^{-J(\lambda_2)t} & \\ 0 & & \ddots \; e^{-J(\lambda_n)t} \end{bmatrix} \begin{bmatrix} - & \mathbb{1}^T/\sqrt{n} & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \longrightarrow \frac{\mathbb{1}}{\sqrt{n}} \frac{\mathbb{1}^T}{\sqrt{n}}$$

Thus, we can conclude that $J(0)=0$, i.e., 0 has algebraic multiplicity one.

Thus, if $L(D)v=0 \Rightarrow v \in \operatorname{span}\{\mathbb{1}\} \Rightarrow \dim \mathcal{N}(L(D)) = 1$

$$\Rightarrow \operatorname{rank}(L(D)) = n-1$$

[Prop 3.8] $\Rightarrow$ D has a rooted out-branching

$\Rightarrow$ G is connected. ∎