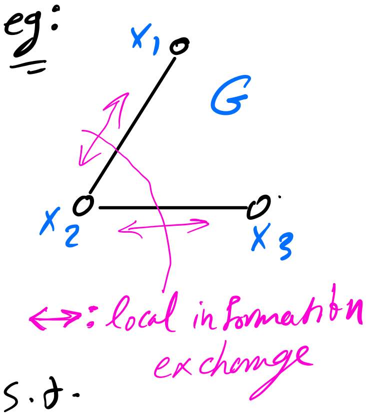
Agreement protocol (consensus)

Let us assign a scalar state "x_i" to each node i in G (undirected).

eg:



N_i : set of nodes adjacent to "i".

$x := [x_1 \ldots x_n]^T \in \mathbb{R}^n$ concatenation of states

Goal : design an update rule for each x_i s.t.

- all x_i converge to an "agreement".
- it only uses information "locally"

First-order agreement protocol:

Suppose each node implements the following first-order dynamics

$$\dot{x}_i(t) = \sum_{j \in N_i} \left(x_j(t) - x_i(t) \right) \quad \text{for } i = 1, \ldots, n.$$

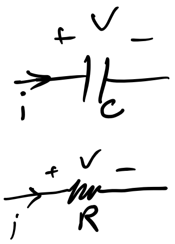
eg.
$$\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 = x_2 - x_3 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = - \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{D(G)} x + \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{A(G)} x$$

so, this can be compactly represented as

$$\dot{x}(t) = -[\underbrace{D(G)}_{\text{degree matrix}} - \underbrace{A(G)}_{\text{adjac. matrix}}]\, x(t) = - \underbrace{L(G)}_{\text{Laplacian matrix}}\, x(t)$$

Circuit interpretation:

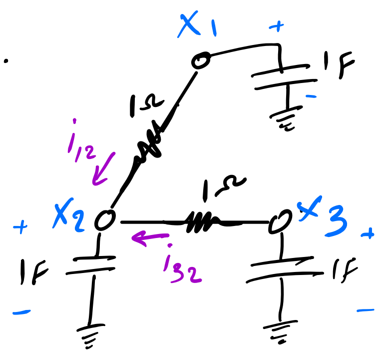
- replace the edges with unit resistors
- Connect a linear unit capacitor from each node to "ground".
- let each $x_i(0)$ denote the initial capacitor charge at node i.



$i = C\dfrac{dv}{dt}$

$V = iR$

e.g.



Kirchhoff's current law at node 2:

$$1.\frac{dx_2}{dt} = i_{12} + i_{32}$$
$$= (x_1 - x_2).1 + (x_3 - x_2).1$$

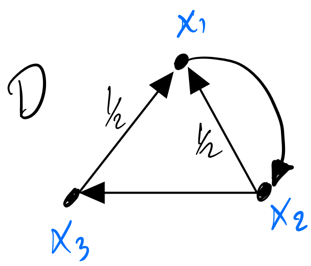
So, kirchhoff's current-voltage law at node "i" implies:

$$\dot{x}_i(t) = \sum_{j \in N_i} \left(x_j(t) - x_i(t)\right)$$

but this is the same as our agreement protocol.

Q: Now, what if the information network is directed (D) ?!

E.g.



Agreement protocol:

$$\begin{cases} \dot{x}_1 = \frac{1}{2}(x_3 - x_1) + \frac{1}{2}(x_2 - x_1) \\ \dot{x}_2 = x_1 - x_2 \\ \dot{x}_3 = x_2 - x_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\underbrace{\begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}}_{\Delta(D)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A(D)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(in)degree matrix of D.

in-degree adjacency matrix of D

$\Rightarrow$ Agreement protocol for directed graph:

$$\dot{x} = -(\Delta(D) - A(D))x = -L(D)x$$

in-degree Laplacian matrix of D.

Q: Are these procedures actually working towards an agreement?

Define: the "Agreement set" $\mathcal{A} \subseteq \mathbb{R}^n$ is the subspace $\text{span}\{\mathbf{1}\}$

i.e.

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid x_i = x_j, \forall i, j\}.$$

Note that both agreement protocols are stationary on the agreement set $\mathcal{A}$. why?

recall that $L(G)\mathbf{1} = 0$ and $L(D)\mathbf{1} = 0$.

Q: Do these protocols actually converge to the agreement set $\mathcal{A}$?!

Yes, but conditionally!

Undirected Network G:

(I) $\dot{x}(t) = -L(G)x$ where $x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$ is prescribed.

- Recall that if G is connected then eigenvalues of $L(G)$ satisfies

$$0 = \lambda_1(G) < \lambda_2(G) \leq \cdots \leq \lambda_n(G)$$

- Recall the solution to the first order linear differential equation (I)

$$x(t) = e^{-L(G)t} x(0)$$

where we can compute $e^{-L(G)t}$ as follows:

$L(G) = U \Lambda(G) U^T$ is the EVD of $L(G)$. and

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}, \quad \Lambda(G) = \begin{bmatrix} \lambda_1(G) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(G) \end{bmatrix}.$$

$$\Rightarrow x(t) = e^{-U\Lambda(G)U^T t} x_0$$

$$= U e^{-\Lambda(G)t} U^T x_0$$

$u_i \perp u_j$

$$= \sum_{i=1}^{n} e^{-\lambda_i(G)t} u_i u_i^T x_0 = \sum_{i=1}^{n} \left[e^{-\lambda_i(G)t} (u_i^T x_0) \right] u_i$$

But note that $u_1 = 1/\sqrt{n}$ (why?), and as $\lambda_i(G) > 0 \;\; \forall i \geq 2$, we can conclude that $(\lambda_1(G) = 0)$: as $t \to \infty$,

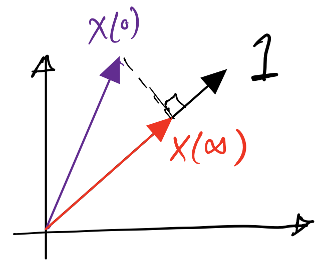
$$x(t) \longrightarrow (u_1^T x_0)\, u_1 = \frac{(1^T x_0)}{n} 1 .$$

Thm: Agreement protocol (I) converges to the agreement set $\mathcal{A}$ if G is connected (with the rate $\lambda_2(G)$).

- Note that $\lambda_2(G)$ is the smallest positive eigenvalue of $L(G)$.

Properties:

1) The convergence point interpretation:



$$\underset{x \in \mathcal{A}}{\text{argmin}} \; \| x - x(0) \| = \text{Proj}_{1} [x(0)] = \frac{1^T x(0)}{1^T 1} \cdot 1 = \left(\frac{1^T x(0)}{n} \right) 1$$

2) $v(t) := 1^T x(t)$ is a constant of motion:

$$\frac{d}{dt} v(t) = 1^T(-L(G)x(t)) = -x^T(t) L(G) 1 = 0 .$$

Q: Now, we know that Connectivity is a sufficient condition for convergence of Agreement Protocol on undirected graphs. Is it also necessary for arbitrary $x(0)$?! Yes, why?

Recall that G is connected iff $\lambda_2(G) > 0$.

- Also, G is connected iff it has a spanning tree.

Connected spanning subgraph with no cycles.

Containing all original nodes

So, having spanning tree is the minimal nec. and suff. condition for convergence of (I).

Q: what happens if G is not connected ?!

Consider G with exactly two connected components. Given x_0, does (I) converges? if yes, can we characterize its limit points, and how they are related to x_0? How about its convergence rate?

[Hint: $\lambda_2 = 0$ but $\lambda_3 > 0$. Find the corresponding eigenvectors and explore similar analysis to the connected case!]

Now, assume D is a directed graph, then.

Directed AP (DAP): $\dot{x}(t) = -L(D)x$.

with $L(D) = \Delta_{in}(D) - A_{in}(D)$. .

Similar to the undirected version, we want to understand the limit set and convergence behavior of this dynamic.

Limit set:

Notice, $\mathbb{1} \in \mathcal{N}(L(D)) \Leftrightarrow \mathbb{1} \subseteq$ Limit set of DAP dynamics

what about the inverse inclusion ?! (more complicated than undirected version)

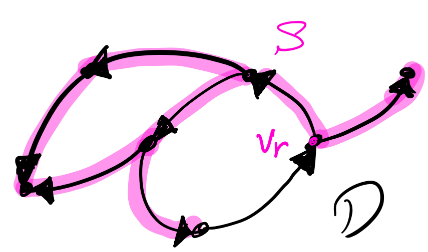
Def: A digraph D is rooted out-branching if

(1) it does not contain a directed cycle

(2) it has a node v_r (root) s.t.

for any other node $v \in D$, $\exists$ a path from v_r to v.

- non-example: D
- example: subgraph S



Proposition (Prop. 3.8 [Meshahi '10])

A digraph $\mathcal{D}$ on n verticies containes a rooted out-branching as a subgraph iff $\text{rank}(L(\mathcal{D})) = n-1$.

In this case, $\mathcal{N}(L(\mathcal{D})) = \mathcal{A}$.

Proof: It suffices to show that "0" has algebraic multiplicity one iff $\mathcal{D}$ contains a rooted out-branching as a subgraph.

why?

Because we know that $\mathbf{1} \in \mathcal{N}(L(\mathcal{D}))$ therefore $\text{rank}(L(\mathcal{D})) \le n-1$, with equality iff 0 is a simple eigenvalue

Now, characteristic polynomial of $L(\mathcal{D})$:

$$P(\lambda) = \det(\lambda I - L(\mathcal{D}))$$
$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda_1 + a_0$$

where a_{n-k} = sum of all principal minors of $L(\mathcal{D})$ of size k.

- But $a_0 = \det(L(\mathcal{D})) = 0$.
- Thus, $\text{rank}\, L(\mathcal{D}) = n-1$ iff $a_1 \neq 0$.
- But, $a_1 = \sum_{v \in \mathcal{D}} \det L_v(\mathcal{D})$ where $L_v(\mathcal{D})$ is a principal submatrix of $L(\mathcal{D})$ with removing the row and the column corresponding to node v.

therefore we need to understand $\det L_v(D)$.

Thm [matrix-Tree theorem (undirected graph G)]:

$$\det L_v(G) = \text{number of spanning trees in } G.$$

Thm [matrix-Tree theorem (digraph D)]:

$$\det L_v(D) = \sum_{T \in \mathcal{T}_v} \prod_{e \in T} w(e)$$

the set of spanning v out-branching subgraphs

Back to the proof:

So, $\det L_v(D) \neq 0$ iff $\exists$ a v-rooted outbranching subgraph of D.

Thus, $a_1 = \sum_v \det L_v(D) \neq 0$ iff $\exists$ a rooted outbranching subgraph of D. ▤

Thus, $\mathcal{A} = \text{span}\{\mathbb{1}\} \subseteq \mathcal{N}(L(D))$

$\not\supseteq$ not true in general but it is true if D has a rooted out-branching subgraph.

Convergence and transient behavior of DAP:

we need to understand the spectrum (eigenvalues) of $L(D)$,

Recall that for undirected graph G;

$L(G)$ is P.S.D. $\Rightarrow \lambda_i \geq 0 \quad \forall i$

but here $L(D)$ is not even symmetric. $\Rightarrow$ it has complex eigenvalues.

Proposition [prop. 3.10 [mesbahi '10]]:

Let $\bar{d}_{in}(D)$ denote the maximum (weighted) in-degree in D.

Then, the spectrum of $L(D)$ lies in

$$\{ z \in \mathbb{C} \mid |z - \bar{d}_{in}(D)| \leq \bar{d}_{in}(D) \};$$

i.e. all its eigenvalues have non-negative real parts.

Proof: It's the direct application of Geršgorian Disk Theorem.

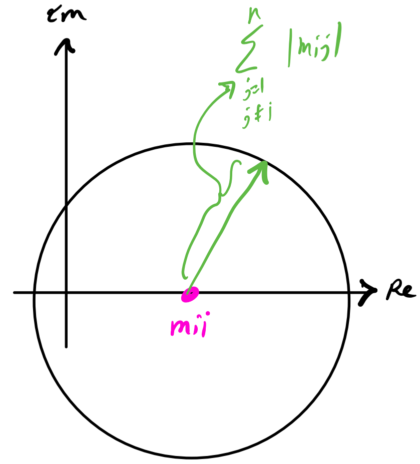
Recall: $M = [m_{ij}]$ is $n \times n$ real matrix. the spectrum of M lies in

$$G(A) := \bigcup_{i=1}^{n} \{ z \in \mathbb{C} \mid |z - m_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |m_{ij}| \}.$$

"row sum" (except diag.)

pictorially:

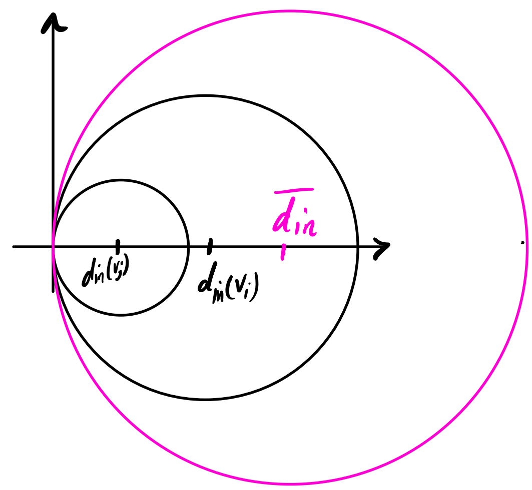
$$\begin{array}{cc} & \text{column } i \\ \text{row } i & \begin{bmatrix} m_{i1} & \cdots & m_{i(i-1)} & \boxed{m_{ii}} & m_{i(i+1)} & \cdots & m_{in} \end{bmatrix} \end{array}$$



Therefore, spectrum of $L(D)$ lies in

$$\bigcup_i \{ z \in \mathbb{C} \mid |z - d_{in}(v_i)| \le d_{in}(v_i) \} ;$$

i.e.



Question: Note that eigenvalues of A and A^T are the same (why?!).
Can you argue why the Geršgorian theorem is also valid for the column sum (except diagonal)?!
Can we claim that the spectrum of A lies in $G(A) \cap G(A^T)$?
What does this mean for $L(D)$?

Now, to understand the solution of DAP we need to compute

$e^{-L(D)t}$!

Consider the Jordan decomposition of $L(D) = P\,J(D)\,P^{-1}$

with $J(D) = \begin{bmatrix} J(0) & & 0 \\ & J(\lambda_2) & \\ & & \ddots & \\ 0 & & & J(\lambda_K) \end{bmatrix}$, $P = [P_1 \cdots P_n]$

If D has a rooted out-branching as its subgraph, then $J(0)=0$!

Also, $L(D)P = PJ(D) \Rightarrow L(D)P_1 = 0 \Rightarrow P_1 \in \text{span}\{\mathbb{1}\}$.

Similarly, $P^{-1}L(D) = J(D)P^{-1} \Rightarrow q_1^T$ (first row of P^{-1}) and $q_1^T L(D) = 0$.

Finally, as $P^{-1}P = I \Rightarrow q_1^T P_1 = 1$.

Now,

$$e^{-L(D)t} = P\,e^{-J(D)t}\,P^{-1} = P \begin{bmatrix} e^{0} & & & \\ & e^{J(-\lambda_2)} & & 0 \\ & 0 & \ddots & \\ & & & e^{J(-\lambda_K)} \end{bmatrix} P^{-1}$$

Now, as $\lambda_2, \dots \lambda_K$ have non-negative real part, we conclude that

$$\lim_{t\to\infty} e^{-L(D)t} = P_1 q_1^T$$ matrix

Thm: If D has a rooted out-branching subgraph, the DAP converges as

$$\lim_{t \to \infty} x(t) = (p_1 q_1^T) x_0$$

where p_1, q_1 are the right and left eigenvectors associated with eigenvalue 0, s.t., $p_1^T q_1 = 1$.

Therefore, $x(t) \to \mathcal{A}$ iff D has a rooted out-branching.

Proof: Recall that $p_1 \in \text{span}\{\mathbb{1}\}$, then choose $p_1 = \mathbb{1}$.

then $x(t) \longrightarrow \mathbb{1} q_1^T x_0 = \underbrace{(q_1^T x_0)}_{\leftarrow \text{ scalar.}} \mathbb{1}$

where $q_1^T \mathbb{1} = 1$.

Note: $q_1^T x$ is the constant of motion. (Differentiate!)

Note: