Review:

Undirected graph $G = (\overset{\text{nodes}}{V}, \underset{\text{undirected edges}}{E})$

Degree matrix of G: (diagonal)

$$\Delta(G) = \begin{pmatrix} d(v_1) & & & 0 \\ & d(v_2) & & \\ & & \ddots & \\ 0 & & & d(v_n) \end{pmatrix}$$

Adjacency matrix of G:

(symmetric) $A(G) = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 & \text{if } \overset{\text{unordered}}{v_i v_j} \in E \\ 0 & \text{otherwise} \end{cases}$

Incidence matrix G (undirected):

Pick an arbitrary direction for each edge in G to get a directed graph G°, then:

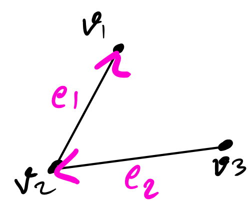
(nodes x edges) $D(G^\circ) = [d_{ij}]$ where $d_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is the tail of } e_j \\ 1 & \text{if } v_i \text{ is the head of } e_j \\ 0 & \text{o.w.} \end{cases}$

graph Laplacian of G: $L(G) \triangleq \Delta(G) - A(G) \overset{(\text{why?})}{=} D(G^\circ) D(G^\circ)^T$

weighted graph Laplacian: $L_w(G) = D(G^\circ) W D(G^\circ)^T$ where $W = \mathrm{diag}(w(e_1), \dots, w(e_m))$

Example:

$$\Delta(G_1) = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad A(G_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$



$G_1 \longleftrightarrow G_1^\circ$

$$D(G_1^\circ) = \begin{array}{c} \begin{matrix} e_1 & e_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \end{array}, \quad L(G_1) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = D(G_1^\circ) D(G_1^\circ)^T \leftarrow (\text{sym.})$$

If $w(e_1) = 5$, $w(e_2) = 7$ $\Rightarrow$

$$L_w(G_1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 12 & -7 \\ 0 & -7 & 7 \end{bmatrix}.$$

Can you guess how to define "weighted Adjacency/degree" matrices for G (undirected) ?!

Directed graph $D = (V, E)$ — V: nodes. — E: directed edges.

Incidence matrix of D:

since D already has orientation $\Rightarrow D(D) = [d_{ij}]$

Adjacency matrix of D with edge weights w_{ij} on edge $(\overrightarrow{v_j, v_i})$:

(not symmetric) $A(D) \triangleq [\vec{a}_{ij}]$ where $\vec{a}_{ij} = \begin{cases} w_{ij} & \text{if } (\overrightarrow{v_j, v_i}) \in E \\ 0 & \text{o.w.} \end{cases}$ ordered

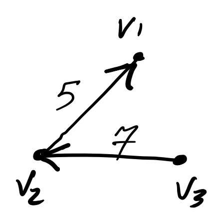
weighted in-degree matrix of D:

$$\Delta(D) = \text{diag}\big(d_{in}(v_1), \dots, d_{in}(v_n)\big) \text{ with } d_{in}(v_i) = \sum_{\{j \mid (\overrightarrow{v_j, v_i}) \in E(D)\}} w_{ij}$$

weighted in-degree Laplacian of D: (not symmetric) $L(D) \triangleq \Delta(D) - A(D)$

Example:

$$\Delta(D_1) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(D_1) = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix},$$



D_1

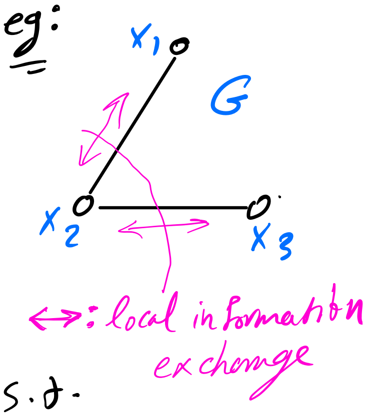
incidence matrix

$$D(D_1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad L(D_1) = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$ (not sym.)

Agreement protocol (consensus)

Let us assign a scalar state "x_i" to each node i in G (undirected).

eg:



N_i: set of nodes adjacent to "i".

$x := [x_1 \ldots x_n]^T \in \mathbb{R}^n$ concatenation of states

Goal: design an update rule for each x_i s.t.

- all x_i converge to an "agreement".
- it only uses information "locally"

First-order agreement protocol:

Suppose each node implements the following first-order dynamics

$$\dot{x}_i(t) = \sum_{j \in N_i} \left(x_j(t) - x_i(t) \right) \qquad \text{for } i = 1, \ldots, n.$$

eg.

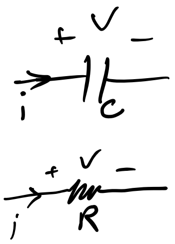
$$\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 = x_2 - x_3 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{D(G)} x + \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{A(G)} x$$

So, this can be compactly represented as

$$\dot{x}(t) = -[\underbrace{D(G)}_{\text{degree matrix}} - \underbrace{A(G)}_{\text{adjac. matrix}}]\, x(t) = -\underbrace{L(G)}_{\text{Laplacian matrix}}\, x(t)$$

Circuit interpretation:

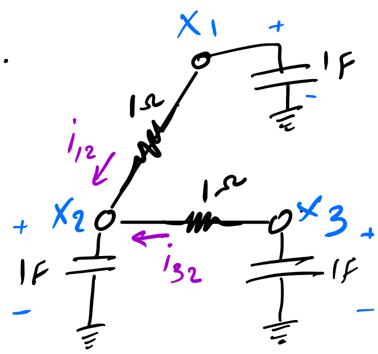
- replace the edges with unit resistors
- Connect a linear unit capacitor from each node to "ground".
- let each $x_i(0)$ denote the initial capacitor charge at node i.



$$i = C\frac{dv}{dt}$$

$$V = iR$$

e.g.



Kirchhoff's current law at node 2:

$$1.\frac{dx_2}{dt} = i_{12} + i_{32}$$
$$= (x_1 - x_2).1 + (x_3 - x_2).1$$

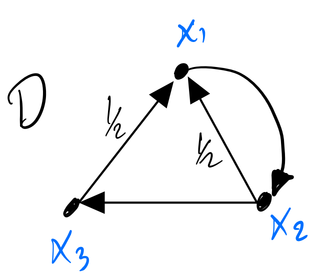
So, kirchhoff's current-voltage law at node "i" implies:

$$\dot{x}_i(t) = \sum_{j\in N_i} \left(x_j(t) - x_i(t)\right)$$

but this is the same as our agreement protocol.

Q: Now, what if the information network is directed (D) ?!

E.g.



Agreement protocol:

$$\begin{cases} \dot{x}_1 = \frac{1}{2}(x_3 - x_1) + \frac{1}{2}(x_2 - x_1) \\ \dot{x}_2 = x_1 - x_2 \\ \dot{x}_3 = x_2 - x_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \end{bmatrix} = - \underbrace{\begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}}_{\Delta(D)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A(D)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(in)degree matrix of D.

in-degree adjacency matrix of D

$\Rightarrow$ Agreement protocol for directed graph:

$$\dot{x} = -\left(\Delta(D) - A(D)\right)x = -L(D)\,x$$

in-degree Laplacian matrix of D.

Q: Are these procedures actually working towards an agreement?

Define: the "Agreement set" $\mathcal{A} \subseteq \mathbb{R}^n$ is the subspace $\text{span}\{\mathbb{1}\}$

i.e.

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid x_i = x_j, \forall i,j\}.$$

Note that both agreement protocols are stationary on the agreement set $\mathcal{A}$. why?

recall that $L(G)\mathbb{1} = 0$ and $L(D)\mathbb{1} = 0$.