Lecture 9

Monday, February 2, 2026 12:27 PM

Review:

Thm: Let $g(x,t) \in C^2(\mathbb{R} \times [0,\infty))$ twice continuously differentiable

Let X_t be an Itô process satisfying $dX_t = u\,dt + v\,dW_t$.

Then, $Y_t = g(X_t, t)$ is Itô process and

$$dY_t = \frac{\partial g}{\partial t}(X_t,t)\cdot dt + \frac{\partial g}{\partial x}(X_t,t)\cdot dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)\cdot (dX_t)^2$$

where $(dX_t)^2 = dX_t \cdot dX_t$ is computed according to the rules

$$dt\cdot dt = dt\cdot dW_t = dW_t\cdot dt = 0, \quad dW_t\cdot dW_t = dt.$$

Examples: $Y_t = \frac{1}{2}W_t^2$ (meaning $g(x,t) = \frac{1}{2}x^2$)

$$dY_t = 0\cdot dt + W_t\,dW_t + \frac{1}{2}\cdot 1\cdot (dW_t)^2 = W_t\,dW_t + \frac{1}{2}dt$$

which is $\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$!

Example: $Y_t = t\cdot W_t$ (meaning $g(x,t) = t\cdot x$)

$$dY_t = W_t\cdot dt + t\,dW_t + 0\cdot (dW_t)^2$$

or: $tW_t = \int_0^t W_s\,ds + \int_0^t s\,dW_s$

or: $\int s\,dW_s = tW_t - \int_0^t W_s\,ds$ (Integration by parts)

Thm: (Integration by parts)

Supp. for a.a ω: $f(\omega, s)$ is continuous with bounded variations in $s \in [0,t]$

Then $$\int_0^t f(s)\,dW_s = f(t)W_t - \int_0^t W_s\,df_s$$

proof : $f(s)\,dW_s + W_s\,df_s \overset{\Downarrow}{=} d(\underbrace{f(s)W_s}) \rightarrow Y_s = g(W_s, s)$ with $g(x,s) = x.f(s)$

Itô formula

so $\frac{\partial^2}{\partial x^2} g = 0$

Multidimensional Itô formula :

Let $d\vec{X}_t = \vec{u}\,dt + v\,d\vec{B}_t$ be n-dim Itô process where

$$\vec{X}_t = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \\ v_{n1} & \cdots & v_{nm} \end{bmatrix}, \quad d\vec{B}_t = \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_n(t) \end{bmatrix}$$

Let $g(x,t) : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}^p$ be C^2 map.

Then $Y(w,t) = g(X_t, t) = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}$ is an Itô process with

$$dY_k = \frac{\partial g_k}{\partial t}(X_t, t)\,dt + \sum_i \frac{\partial g_k}{\partial x_i}(X_t, t)\,dx_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X_t, t)\,dx_i \cdot dx_j$$

where $dW_i \cdot dW_j = \delta_{ij}\,dt$, $dW_i \cdot dt = dt \cdot dW_i = 0$.

Proof of Itô Formula : (read Øksendal pp. 46-48)

In 1-dim, we can rewrite it as follows: $Y_t = g(X_t, t)$

$$dY_t = \frac{\partial g}{\partial t}(X_t, t) \cdot dt + \frac{\partial g}{\partial x}(X_t, t) \cdot dX_t + \frac{1}{2}\frac{\partial g^2}{\partial x^2}(X_t, t) \cdot (dX_t)^2$$

but $dX_t = u_t\,dt + v_t\,dW_t$ so, we need to show that

$$dY_t = \left[\frac{\partial g}{\partial t}(X_t, t) + \frac{\partial g}{\partial x}(X_t, t) \cdot u_t + \frac{1}{2}\frac{\partial g^2}{\partial x^2}(X_t, t)\, v_t^2\right] dt + \frac{1}{2}\frac{\partial g}{\partial x}(X_t, t)\, v_t\,dW_t$$

It suffices to only show this for the case where $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ are bounded and u, v are elementary r.v.s: Given a fine enough partition Π of $[0,T]$ recall our notation $\Delta t_i = t_{i+1} - t_i$, $\Delta X_i = X_{t_{i+1}} - X_{t_i}$, $\Delta g_j = g(X_{t_{i+1}}, t_{i+1}) - g(X_{t_i}, t_i)$ and consider the Taylor expansion:

$$g(X_t, t) = g(X_0, t_0 = 0) + \sum_i \Delta g_i$$

$$\text{(Taylor exp.)} = g(X_0, 0) + \sum_i \frac{\partial g}{\partial t} \Delta t_i + \sum_i \frac{\partial g}{\partial x} \Delta X_i$$
$$+ \frac{1}{2} \sum_i \left(\frac{\partial^2 g}{\partial t^2} (\Delta t_i)^2 + 2 \frac{\partial^2 g}{\partial t \partial x} (\Delta t_i)(\Delta X_i) + \frac{\partial^2 g}{\partial x^2} (\Delta X_i)^2 \right)$$
$$+ HoT(\|\Delta t_i\|^2, \|\Delta X_i\|^2)$$

Now, as Π gets finer $(\Delta t_i \to 0)$:

$$\sum_i \frac{\partial g}{\partial t} \Delta t_i \to \int \frac{\partial g}{\partial t}(X_s, s)\, ds$$ first term ✓

$$\sum_i \frac{\partial g}{\partial x} \Delta X_i \to \int \frac{\partial g}{\partial x}(X_s, s)\, dX_s$$ second term ✓

as u, v are elementary: $\Delta X_i = u_i \Delta t_i + v_i \Delta W_i$ so

$$(\Delta X_i)^2 = u_i^2 (\Delta t_i)^2 + 2 u_i v_i (\Delta t_i) \cdot (\Delta W_i) + v_i^2 (\Delta W_i)^2$$

the third term:

$$= \frac{1}{2} \sum_i \left[\boxed{}_i (\Delta t_i)^2 + \boxed{}_i (\Delta t_i) \cdot (\Delta W_i) + \frac{\partial^2 g}{\partial x^2} v_i^2 (\Delta W_i)^2 \right]$$

because $\mathbb{E}[\Delta W_i \Delta W_j] = \delta_{ij}$:

$$\mathbb{E}\left[\left(\sum_i \Box_i (\Delta t_i)(\Delta W_i) - 0 \right)^2 \right] = \sum_i \mathbb{E}[\Box_i^2] \underbrace{\mathbb{E}[\Delta W_i^2]}_{\Delta t_i} \cdot (\Delta t_i)^2 = \mathcal{O}(\|\Delta t_i\|^3)$$

So : as $|\Delta t_i| \to 0$, then $\sum_i \square_i (\Delta t_i)^2$ or $\sum_i \square_i (\Delta t_i) \cdot (\Delta W_i) \to 0$ in L^2.

Last claim: $\sum_i \underbrace{\frac{\partial^2 g}{\partial x^2} v_i^2}(\Delta W_i)^2 \longrightarrow \int \frac{\partial^2 g}{\partial x^2} v^2 \, dt$ in $L^2(\mathbb{P})$!

Consider any $\sum_i a_i (\Delta W_i)^2$ and compute

$$\mathbb{E}\left[\left(\sum_i a_i (\Delta W_i)^2 - \sum_i a_i \Delta t_i\right)^2\right] = \sum_{i,j} \mathbb{E}\left[a_i a_j (\Delta W_i^2 - \Delta t_i)(\Delta W_j^2 - \Delta t_j)\right]$$

$$\begin{cases} \text{if } i<j : & a_i a_j (\Delta W_i^2 - \Delta t_i) \perp \Delta W_j^2 - \Delta t_j \longrightarrow \mathbb{E} = 0 \\ \text{if } j>i : & a_i a_j (\Delta W_j^2 - \Delta t_j) \perp \Delta W_i^2 - \Delta t_i \to \mathbb{E} = 0 \end{cases}$$

$$= \sum_i \mathbb{E}\left[a_i^2 (\Delta W_i^2 - \Delta t_i)^2\right]$$

$$= \sum_i \mathbb{E}\left[a_i^2 (\Delta W_i^4 - 2 \Delta W_i^2 \Delta t_i + \Delta t_i^2)\right]$$

$(a_i \perp \Delta W_i)$ $$= \sum_i \mathbb{E}[a_i^2] \cdot \mathbb{E}\left[\Delta W_i^4 - 2 \Delta W_i^2 \Delta t_i + \Delta t_i^2\right]$$

$$= \sum_i \mathbb{E}[a_i^2] \left(3 \Delta t_i^2 - 2 \Delta t_i^2 + \Delta t_i^2\right)$$ ← 2nd. and 4th moment of $\Delta W_i \sim N(0, \Delta t_i)$

$$= 2 \sum_i \mathbb{E}[a_i^2] \Delta t_i^2 \longrightarrow 0 \quad \text{as } |\Delta t_i| \to 0 .$$

In summary $\sum_i a_i (\Delta W_i)^2 \to \int a_s \, ds$ in $L^2(\mathbb{P})$! ∎

Itô Representation Theorem:

Supp. $\vec{W}_t = (W_t^1, \dots, W_t^n)$ is n-dim Wiener process

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For any $v(\omega,s) \in \mathcal{V}^n(0,T)$, by construction

$$V(\omega) := \int_0^T v(\omega,s).d\vec{W}_s \in \mathcal{L}^2(\mathcal{F}_T^n, \mathbb{P})$$

i.e., is $\mathcal{F}_T^n$-meas and $E[V^2] < \infty$ which by Itô isometry

means $\int_0^T E[v^2]\,dt < \infty$.

Thm: The converse of this is true! (Itô Rep

For any $V \in \mathcal{L}^2(\mathcal{F}_T^n, \mathbb{P})$, $\exists!$ $v(\omega,t) \in \mathcal{V}^n(0,T)$ s.t.

$$V(\omega) - \mathbb{E}[V] = \int_0^T v(\omega,t).dW_t .$$

proof: read [Chp. 4.3, Øksendal]

As a natural extension, we know that

for any $v \in \mathcal{V}^n(0,\infty)$, $X_t = X_0 + \int_0^t v(\omega,s)\,dW_s$, $t \geq 0$

is a martingale w.r.t. $\{\mathcal{F}_t^n\}$ an w.r.t. $\mathbb{P}$.

Thm: The converse of this is true! (Martingale Representation Thm)

Let M_t be an $\mathcal{F}_t^n$-martingale and $M_t \in \mathcal{L}^2(\mathbb{P})$, $\forall t \geq 0$.

$\Rightarrow \exists!$ $g(\omega,t) \in \mathcal{V}^n(0,t)$, $\forall t \geq 0$ s.t.

$$M_t(\omega) - E[M_0] = \int_0^t g(\omega,s)\,dW_s \quad \text{a.s., } \forall t \geq 0.$$

proof: (n=1) by Itô Rep. thm, for each $T=t$, $V=M_t$ we have

$\exists\, v^{(t)}(\omega,s) \in L^2(\mathcal{F}_t, \mathbb{P})$ s.t.

$$M_t(\omega) = \underbrace{\mathbb{E}[M_t]}_{=\mathbb{E}[M_0]} + \int_0^t v^{(t)}(\omega,s)\, dW_s(\omega)$$

$= \mathbb{E}[M_0]$ as M_t is a martingale!

also suppose $0 \le t_1 \le t_2$:

$$M_{t_1} = \mathbb{E}[M_{t_2} \mid \mathcal{F}_{t_1}] = \mathbb{E}[M_0] + \mathbb{E}\left[\int_0^{t_2} v^{(t_2)}(\omega,s)\, dW_s(\omega) \,\Big|\, \mathcal{F}_{t_1}\right]$$

$$= \mathbb{E}[M_0] + \int_0^{t_1} v^{(t_2)}(\omega,s)\, dW_s(\omega)$$

but we also have $M_{t_1} = \mathbb{E}[M_0] + \int_0^{t_1} v^{(t_1)}(\omega,s)\, dW_s(\omega)$.

$$\Rightarrow 0 = \mathbb{E}\left[\left(\int_0^{t_1} (v^{(t_2)} - v^{(t_1)})\, dW_s\right)^2\right] \underset{\text{Itô isometry}}{=} \mathbb{E}\left[\int_0^{t_1} (v^{(t_2)} - v^{(t_1)})^2\, ds\right]$$

$$= \| v^{(t_2)} - v^{(t_1)} \|_{L^2(\mathbb{P}\times\mu)}$$

$\Rightarrow v^{(t_1)} = v^{(t_2)}$ for a.a. $(\omega,t) \in \Omega \times [0,t_1]$.

So just define $g(\omega,s) = v^{(N)}(\omega,s)$ if $s \in [0,N]$ and see

$$\forall t \geq 0:\ M_t = \mathbb{E}[M_0] + \int_0^t v^{(t)}(\omega,s)\, dW_s(\omega) = \mathbb{E}[M_0] + \int_0^t g(\omega,s)\, dW_s(\omega)$$ ∎

Exercises: Øksendal: (4.1), (4.2), (4.3), (4.4), (4.5), (4.6)
(4.8.a), (4.11), (4.12), (4.13), (4.14)
(4.16)