Lecture 8

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<u>Thm:</u> let $f(\omega,t) \in \mathcal{V}(0,T)$ for all T. Then

$$I_t(\omega) = \int_0^t f(\omega,s)\, dW_s$$

is a martingale w.r.t. $\mathcal{F}_t^W$, has continuous path w.p. 1, and

$$P\left[\sup_{0\le t\le T} |I_t| \ge \lambda\right] \le \frac{1}{\lambda^2} \mathbb{E}\left[\int_0^T f(\omega,s)^2 ds\right]; \forall \lambda, T>0.$$

<u>Proof sketch:</u> Consider elementary integrands

$$\varphi_n = \varphi_n(\omega,t) = \sum_j e_j(\omega)\, \chi_{[t_j, t_{j+1})}(t)$$

such that $\varphi_n \to f$ in $L^2(P, \mu_T)$: $E\left[\int_0^T (f-\varphi_n)^2 dt\right] \to 0$

put $I_n(\omega,t) = I(\varphi_n)_{(\omega,t)} = \int_0^t \varphi_n(\omega,s)\, dW_s(\omega)$, and note that

1 - $I_n(\omega,\cdot)$ is continuous because $W_\cdot(\omega)$ is continuous w.p. 1.

2 - $I_n(\omega,t)$ is a martingale w.r.t. $\mathcal{F}_t^W$:

$$\mathbb{E}\left[I_n(\omega,t) / \mathcal{F}_s^W\right] = \mathbb{E}\left[\left(\underbrace{\int_0^s \varphi_n\, dW}_{\in \mathcal{F}_s} + \int_s^t \varphi_n\, dW\right) \Big/ \mathcal{F}_s\right]$$

by def. $$= \int_0^s \varphi_n\, dW + \mathbb{E}\left[\sum_j e_j(\omega)\, \Delta W_j(\omega) / \mathcal{F}_s\right]$$

by tower property $\mathcal{F}_s \subset \mathcal{F}_{t_j}$ $$= \int_0^s \varphi_n\, dW + \sum_{t_j > s}^{t} \mathbb{E}\left[\mathbb{E}\left[e_j\, \Delta W_j / \mathcal{F}_{t_j}\right] / \mathcal{F}_s\right]$$

$$\sum^{t} \mathbb{E}\left[e_j\, \mathbb{E}\left[\Delta W_j / \mathcal{F}_{t_j}\right] / \mathcal{F}_s\right]$$

$e_j \in F_{t_j}$ $\quad = \int_0^s \varphi_n \, dW + \sum_{t_j > s} \mathbb{E}[\cdots] \underbrace{\qquad}_{=0}$

$$= \int_0^s \varphi_n \, dW = I_n(\omega, s) \, . \quad \checkmark$$

Next, to generalize this to any function $f \in \mathcal{V}$, we wanna take the limit $I_n \to I_t$ defined as

$$I_t = I(f)(\omega,t) = \int_0^t f(\omega,s) \, dW_s(\omega) \, , \quad 0 \le t \le T,$$

and argue that I_t is continuous a.s. (up to zero probability modifications) and is martingale: $\mathbb{E}[I_t / F_s] = I_s$.

The next argument is technical [page 33, Øksendal] and involves Borell-Cantelli Lemma and Doob's Martingale Inequality (DMI):

Thm: (DMI) If M_t: martingal with continuous paths ($t \mapsto M_t$ is continuous a.s.)

then $\forall p \ge 1$, $T \ge 0$, $\lambda > 0$

$$\mathbb{P}\Big[\sup_{0 \le t \le T} |M_t| \ge \lambda \Big] \le \frac{1}{\lambda^p} \mathbb{E}\big[|M_T|^p \big] .$$

Here it is: note that $I_n - I_m$ is a matingal, so

$$\mathbb{P}\Big[\sup_{0 \le t \le T} |I_n(t) - I_m(t)| > \varepsilon \Big] \le \frac{1}{\varepsilon^2} \mathbb{E}\big[|I_n(T) - I_m(T)|^2 \big]$$

$$= \frac{1}{\varepsilon^2} \mathbb{E}\Big[\int_0^T (\varphi_n - \varphi_m)^2 \Big]$$

$$\to 0 \text{ as } n,m \to \infty .$$

(passing to subsequence) pick $\varepsilon = 2^{-k}$, chose a subsequence $\{n_k\}$ s.t.

$$\mathbb{P}\Big[\sup_{0 \le t \le T} \big| I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega) \big| > 2^{-k} \Big] < 2^{-k}$$

Now, we have a fast diminishing probability so by Borell-Cantelli lemma:

Borell-Cantelli Lemma:

$$\mathbb{P}\left[\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|>2^{-k},\ \text{infinitely often}\right]=0$$

$\Rightarrow$ for a.a. ω, $\exists\ K_1(\omega)$ s.t.

$$\sup_{0\le t\le T}\left|I_{n_{k+1}}(t,\omega)-I_{n_k}(t,\omega)\right|\le 2^{-k}\quad \forall k\ge K_1(\omega)$$

$\Rightarrow$ For a.a. ω: $I_{n_k}(t,\omega)$ is uniformly convergent for $t\in[0,T]$.

$\to$ denote the limit by $J(t,\omega)$ which is continuous in t, a.s.!

since $I_{n_k}(t,\cdot)\longrightarrow I(t,\cdot)$ in $\mathcal{L}^2(\mathbb{P})$ $\forall t$, we must have $I(t,\cdot)=J(t,\cdot)$ a.s., $\forall t\in[0,T]$.

So, we conclude that $I(t,\omega)$ is continuous in t, a.s.

The last argument: As $I_n\to I$ in $\mathcal{L}^2(\mathbb{P},\mu)$, by DCT

$$\mathbb{E}\left[I_t\,\middle|\,\mathcal{F}_s^W\right]=\mathbb{E}\left[\mathcal{L}^2\text{-}\lim_{n\to\infty} I_n(t)\,\middle|\,\mathcal{F}_s^W\right]$$

$$=\mathcal{L}^2\text{-}\lim_{n\to\infty}\mathbb{E}\left[I_n(t)\,\middle|\,\mathcal{F}_s^W\right]$$

$$=\mathcal{L}^2\text{-}\lim_{n\to\infty} I_n(s)=I_s.\ \Rightarrow I_t \text{ is martingale!}$$

Finally, I_t is a martingale with continuous paths, so by Doob's martingal Inequality:

$$\mathbb{P}\left(\sup_{0\le t\le T}|I_t|>\lambda\right)\le\frac{1}{\lambda^2}\mathbb{E}\left[|I_T|^2\right]$$

$$=\frac{1}{\lambda^2}\mathbb{E}\left[\left(\int_0^T f\,dW_t\right)^2\right]$$

by Itô isometry $$=\frac{1}{\lambda^2}\mathbb{E}\left[\int_0^T f^2\,dt\right]$$

Generalization of Itô process to multi-dimensions:

Let $\vec{W} = (W_1, \dots, W_n)$ be n-dim. Brownian motion

Let $\mathcal{F}_t^{\vec{W}} = \sigma\left(\{ W_k(s) : s \le t, \ k = 1, \dots, n \}\right)$

Let $\mathcal{V}^{m \times n}(S, T)$ be the set of $m \times n$ matrices $[v_{ij}(\omega, t)]$ s.t.

1) $(\omega, t) \mapsto v_{ij}(\omega, t)$ is $\mathcal{F} \times \mathcal{B}$ meas.

2) $v_{ij}(\cdot, t) \in \mathcal{F}_t^{\vec{W}}$, $\forall t$.

3) $\| v_{ij}^2 \|_{\mathcal{L}^2(P \times \mu)} = \mathbb{E}\left[\int_S^T v_{ij}^2(\omega, t)\, dt \right] < \infty$.

Then
$$\int_S^T v \, d\vec{W} = \int_S^T \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & & & \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix} \begin{pmatrix} dW_1 \\ \vdots \\ dW_n \end{pmatrix}$$

with element-wise integral.

Exercises: (Øksendal chap. 3)

1) $\int_0^t s \, dW_s = t W_t - \int_0^t W_s \, ds$ (3.1)

2) $\int_0^t W_s^2 \, dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s \, ds$ (3.2)

3) show $\begin{cases} W_t^2 - t \text{ is martingale} & (3.5) \\ W_t^3 - 3t W_t \text{ is martingale} & (3.6) \end{cases}$

- (3.3) - (3.8) - (3.13) - (3.16) - (3.18)

→ Doob - Dynkin Lemma: $X, Y : \Omega \to \mathbb{R}$

Y is $\sigma(X)$-measurable $\Longleftrightarrow$ $\exists\, g : \mathbb{R} \to \mathbb{R}$ Borel-measu. s.t $Y = g(X)$.

Stochastic Differential Equations:

Ex: you showed that $\int W_s \, dW_s = \frac{1}{2} W_s^2 - \frac{1}{2} t$

So, $\frac{1}{2} W_s^2 = \frac{1}{2} t + \int W_s \, dW$

or: " $d\left(\frac{1}{2} W_s^2\right) = \frac{1}{2} dt + W_s \, dW_s$ "

Def: (1-dim Itô process - diffusion process)

let W_t be Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$

$$X_t = X_0 + \int_0^t u(\omega, s) \, ds + \int_0^t v(\omega, s) \, dW_s$$

or " $dX_t = u \, ds + v \, dW_s$ "

where $\mathbb{P}\left(\int_0^t v(\omega, s)^2 \, ds < \infty, \forall t \geq 0 \right) = 1$

and $\mathbb{P}\left(\int_0^t u(\omega, s)^2 \, ds < \infty, \forall t \geq 0 \right) = 1$

$L^2(\mathbb{P}, \ldots)$ w.r. I. .

for short: $v, u \in \mathcal{L}(\mathbb{R}\times\mu)$ ...

The chain-rule replaced: the 1-dim Itô formula

Thm: let $g(x,t) \in C^2(\mathbb{R}\times[0,\infty))$ twice continuously differentiable

let X_t be an Itô process satisfying $dx_t = u\,dt + v\,dW_t$.

Then, $Y_t = g(X_t, t)$ is Itô process and

$$dY_t = \frac{\partial g}{\partial t}(X_t,t)\cdot dt + \frac{\partial g}{\partial x}(X_t,t)\cdot dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)\cdot(dX_t)^2$$

where $(dX_t)^2 = dx_t \cdot dx_t$ is computed according to the rules

$$dt\cdot dt = dt\cdot dW_t = dW_t\cdot dt = 0\,, \quad dW_t\cdot dW_t = dt.$$

Examples: $Y_t = \frac{1}{2}W_t^2$ (meaning $g(x,t) = \frac{1}{2}x^2$)

$$dY_t = 0\cdot dt + W_t\,dW_t + \tfrac{1}{2}\cdot 1\cdot(dW_t)^2 = W_t\,dW_t + \tfrac{1}{2}dt$$

which is $\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$!

Example: $Y_t = t\cdot W_t$ (meaning $g(x,t) = t\cdot x$)

$$dY_t = W_t\cdot dt + t\,dW_t + 0\cdot(dW_t)^2$$

or: $tW_t = \int_0^t W_s\,ds + \int_0^t s\,dW_s$

or: $\int s\,dW_s = tW_t - \int_0^t W_s\,ds$ (Integration by parts)