Lecture 7

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Conclusion: This bad example in defining $h_n(t_i)$ uses

(I) future information at time t_i, and

(II) looks for convergence uniformly in t.

Resolving these issues for defining $I(\sigma)$:

(I) $\rightarrow$ The first one is easy; we just require

$$\boxed{\sigma_t \in \mathcal{F}_t, \ \forall t}$$

(II) $\rightarrow$ For the second, we look for $\boxed{\text{Convergence in } \mathcal{L}_2}$ because W_t has finite Quadratic variance on $[0,T]$; i.e.

Lemma 1: W_t has finite Quadratic variance on $[0,T]$; i.e.

$$\sum_{t_i \in \Pi_n} (W_{t_{i+1}} - W_{t_i})^2 \xrightarrow{n\to\infty} T \quad \text{in } \mathcal{L}^2$$

Proof:

$$\mathbb{E} \sum_{t_i \in \Pi_n} (W_{t_{i+1}} - W_{t_i})^2 = \sum_{t_i \in \Pi_n} (t_{i+1} - t_i) = T - 0 = T.$$

Now, define the centered r.v. $Z_i = (W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)$ for $t_i \in \Pi_n$

so that $Z_i \perp Z_j$, $i \neq j$. and $Z_i \overset{\text{in law}}{=} (\xi^2 - 1)(t_{i+1} - t_i)$, $\xi \sim \mathcal{N}(0,1)$

Now,

$$\mathbb{E}\left[\left(\sum_{t_i} (W_{t_{i+1}} - W_{t_i})^2 - T\right)^2\right] = \mathbb{E}\left[\sum_{t_i \in \Pi_n} (Z_i)^2\right]$$

$$= \mathbb{E}\left[(\xi^2 - 1)^2\right] \sum_{t_i \in \Pi_n} (t_{i+1} - t_i)^2$$

$$= \mathbb{E}\left[(\xi^2 - 1)^2\right] \cdot T \cdot \sup_{t_i \in \Pi_n} |t_{i+1} - t_i|^2 \longrightarrow 0 \quad \text{as } n \to \infty$$

for any increasingly finer seq of partitions $\{\Pi_n\}$. $\blacksquare$

... : i.e.

Lemma 2: If $\varphi(\omega,t)$ is bounded and elementary,

$$\varphi(\omega,t) = \sum_{i \in \Pi_n} e_i(\omega) \cdot \chi_{[t_i, t_{i+1})}(t), \text{ then}$$

Itô isometry

$$\mathbb{E}\left[\left(\int_0^T \varphi_t \, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T \varphi_t^2 \, dt\right]$$

Proof: set $\Delta W_j = W_{t_{j+1}} - W_{t_j}$, and compute

$$\mathbb{E}\left[e_i e_j \Delta W_i \Delta W_j\right] = \begin{cases} \mathbb{E}\left[e_i^2\right] \cdot (t_{i+1} - t_i) & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$

$\rightsquigarrow \Delta W_i \perp \Delta W_j$

$$\text{So } LHS = \sum_{i,j} \mathbb{E}\left[e_i e_j \cdot \Delta W_i \cdot \Delta W_j\right] = \sum_i \mathbb{E}\left[e_i^2\right](t_{i+1} - t_i) = RHS. \blacksquare$$

Summary (The Itô Integral for real): set W_t on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$

Def: Let $\mathcal{V} = \mathcal{V}(0,T)$ be the class of functions

$$\sigma(\omega,t) : \Omega \times [0,T] \longrightarrow \mathbb{R}$$

such that

I) $(\omega,t) \longmapsto \sigma(\omega,t)$ is $\mathcal{F} \times \mathcal{B}([0,T])$-measurable

II) $\sigma(\omega,t)$ is $\mathcal{F}_t$-adapted

III) $\mathbb{E}\left[\int_0^T \sigma(\omega,t)^2 \, dt\right] < \infty$; i.e., $\sigma \in \mathcal{L}^2(\mathbb{P} \times \mu_{[0,T]})$.

Def: (Itô integral) let $\sigma \in \mathcal{V}(0,T)$ then

$$\int^T \sigma_t \, dW_t \triangleq \int^T \sigma(\omega,t) \, dW_t(\omega) = \mathcal{L}^2\text{-}\lim_{n\to\infty} \int_0^T \varphi_n(\omega,t) \, dW_t(\omega)$$

where $\{\varphi_n\}$ is <u>any</u> seq. of (bounded) elementary functions s.t.

$$\varphi_n \to \sigma \text{ in } \mathcal{L}_2(\mathbb{P} \times \mu(0,T)) \text{ : i.e.,}$$

$$\mathbb{E}\left[\int_0^T \left(f(w,t) - \varphi_n(w,t)\right)^2\right] \to 0 \text{ as } n \to \infty.$$

Thm: 1) For any $\sigma \in \mathcal{V}(0,T)$ there always exists such $\{\varphi_n\}$.

[page 27-29 in Øksendal]

2) the limit exists and the value of integral doesn't depend on the choice of $\{\varphi_n\}$.

3) the Itô isometry holds for any $\sigma \in \mathcal{V}(0,T)$.

4) If $\sigma_n \in \mathcal{V}(0,T)$ is any seq. s.t. $\sigma_n \to \sigma \in \mathcal{V}(0,T)$ in $\mathcal{L}^2(\mathbb{P}, \mu(0,T))$, then $I(\sigma_n) \to I(\sigma)$ in $\mathcal{L}^2(\mathbb{P})$.

Proof: 1) show it in three approximation steps in $\mathcal{L}^2(\mathbb{P} \times \mu(0,T))$ and using Dominated Convergence theorem:

i) any $\sigma \in \mathcal{V}$ can be approximated by a bounded sequence $\{h_n\} \subset \mathcal{V}$:

$$h_n(w,t) = \begin{cases} \sigma(w,t) & |\sigma(w,t)| < n \\ -n & \sigma(w,t) < -n \\ n & \sigma(w,t) > n. \end{cases}$$

ii) any bound $h \in \mathcal{V}$ can be approximated by a bounded sequence $\{g_n\} \subset \mathcal{V}$ s.t. $g_n(w, \cdot)$ is continuous for all w and n; (harder to show)

$$g_n(t,w) \triangleq \psi_n(t) * h(t,w)$$

... ψ_n is a seq of cont. non-neg. functions where ψ_n ...

s.t. $\int_{-\infty}^{+\infty} \psi_n(t)\, dt = 1$ and $\psi_n(t) \to \delta(t)$.

iii) any $g \in \mathcal{V}$ bounded and $g(\omega, \cdot)$ continuous for each ω, can be approximated by bounded elementary functions $\{\varphi_n\} \subset \mathcal{V}$: $\varphi_n(\omega, t) = \sum_j g(\omega, t_j)\, \chi_{[t_j, t_{j+1})}(t)$.

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2) Use the Itô isometry for elementary integrands.

3 and 4) " " " " " " , and the definition of $\int_0^T \sigma_t\, dW_t$.

Exercise : Complete the proof above!

Exercise : Show
$$\int_0^t W_s\, dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t .$$

(Hint : Choose $\varphi_n(\omega, t) = \sum_{t_i \in \Pi_n} W_i(\omega) \cdot \chi_{[t_i, t_{i+1})}(t)$ and

show that LHS $= \lim_{\Delta t_i \to 0} \sum_i W_i\, \Delta W_i$ with $\Delta W_i = W_{t_{i+1}} - W_{t_i}$.

Next, show that $W_t^2 = \sum_i \Delta(W_i^2) = \sum_i (\Delta W_i)^2 + 2 \sum_i W_i\, \Delta W_i$.

Finally, argue that $\sum_i (\Delta W_i)^2 \longrightarrow t$ in $\mathcal{L}^2(\mathbb{P})$, and conclude!)

Note : $\int_0^t W_s\, dW_s$ does not follow the chain rule!

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more Properties of Itô integral:

Thm: let $f, g \in \mathcal{V}(0,T)$, and $0 \leq S < U < T$. then

5) $\int_S^T f\,dW_t = \int_S^U f\,dW_t + \int_U^T f\,dW_t$ a.s. (a.a. ω)

6) $\int_S^T (cf+g)\,dW_t = c\int_S^T f\,dW_t + \int_S^T g\,dW_t$

7) $\mathbb{E}\left[\int_S^T f\,dW_t\right] = 0$

8) $\int_S^T f\,dW_t$ is $\mathcal{F}_T^W$-measurable.

Recall: Itô Isometry: $\mathbb{E}\left[\left(\int_S^T f\,dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2\,dt\right]$

Proof: show each one for elementary functions, then justify the limit remains true for all $f, g \in \mathcal{V}(0,T)$.

- The last important property of Itô Integral is that it is a martingale:

Thm: let $f(\omega,t) \in \mathcal{V}(0,T)$ for all T. Then

$$I_t(\omega) = \int_0^t f(\omega,s)\,dW_s$$

is a martingale w.r.t. $\mathcal{F}_t^W$, and has continuous path w.p. 1

$$P\left[\sup_{0\le t\le T} |I_t| \geq \lambda\right] \leq \frac{1}{\lambda^2}\mathbb{E}\left[\int_0^T f(\omega,s)^2\, ds\right] ;\ \forall \lambda, T>0.$$

<u>Proof sketch:</u> Consider elementary integrands

$$\varphi_n = \varphi_n(\omega,t) = \sum_j e_j(\omega)\, \chi_{[t_j, t_{j+1})}(t)$$

such that $\varphi_n \to f$ in $\mathcal{L}^2(\mathbb{P}, \mu_T)$: $E\left[\int_0^T (f-\varphi_n)^2 dt\right] \to 0$

put $I_n(\omega,t) = I(\varphi_n)_{(\omega,t)} = \int_0^t \varphi_n(\omega,s)\, dW_s(\omega)$, and note that

- $I_n(\omega,\cdot)$ is continuous because $W_\cdot(\omega)$ is continuous w.p. 1.
- $I_n(\omega,t)$ is a martingale w.r.t. $\mathcal{F}_t^W$.

$$\mathbb{E}\left[I_n(\omega,t) \,/\, \mathcal{F}_s^W\right] = \mathbb{E}\left[\left(\underbrace{\int_0^s \varphi_n\, dW}_{\in \mathcal{F}_s} + \int_s^t \varphi_n\, dW\right) \Big/ \mathcal{F}_s\right]$$

by def. $\quad = \int_0^s \varphi_n\, dW + \mathbb{E}\left[\sum_j e_j(\omega)\, \Delta W_j(\omega) \,/\, \mathcal{F}_s\right]$

by tower property $\mathcal{F}_s \subset \mathcal{F}_{t_j}$ $\quad = \int_0^s \varphi_n\, dW + \sum_{t_j > s}^{t} \mathbb{E}\left[\mathbb{E}\left[e_j\, \Delta W_j \,/\, \mathcal{F}_{t_j}\right] / \mathcal{F}_s\right]$

$e_j \in \mathcal{F}_{t_j}$ $\quad = \int_0^s \varphi_n\, dW + \sum_{t_j > s}^{t} \mathbb{E}\left[e_j \underbrace{\mathbb{E}\left[\Delta W_j \,/\, \mathcal{F}_{t_j}\right]}_{=0} / \mathcal{F}_s\right]$

$$= \int_0^s \varphi_n\, dW = I_n(\omega,s). \quad \checkmark$$

Next, to generalize this to any function $f \in \mathcal{V}$, we wanna take the limit $I_n \to I_t$ defined as

$$I_t = I(f)(\omega, t) = \int_0^t f(\omega, s)\, dW_s(\omega), \quad 0 \le t \le T,$$

and argue that I_t is continuous a.s. (up to zero probability modifications) and is martingale: $\mathbb{E}[I_t \mid \mathcal{F}_s] = I_s$.

The next argument is technical [page 33, Øksendal] and involves Borell-Candelli Lemma and Doob's Martingale Inequality:

Thm: (DmI) If M_t: martingal with continuous paths $t \mapsto M_t$ is continuous a.s.

then $\forall p \ge 1$, $T \ge 0$, $\lambda > 0$

$$\mathbb{P}\left[\sup_{0 \le t \le T} |M_t| \ge \lambda\right] \le \frac{1}{\lambda^p} \mathbb{E}\left[|M_T|^p\right].$$

Here it is: note that $I_n - I_m$ is a matingal, so

$$\mathbb{P}\left[\sup_{0 \le t \le T} |I_n(t) - I_m(t)| > \varepsilon\right] \le \frac{1}{\varepsilon^2} \mathbb{E}\left[|I_n(T) - I_m(T)|^2\right]$$

$$= \frac{1}{\varepsilon^2} \mathbb{E}\left[\int_0^T (\varphi_n - \varphi_m)^2\right]$$

$$\to 0 \text{ as } n, m \to \infty.$$

(passing to subsequence) pick $\varepsilon = 2^{-k}$, chose a subsequence $\{n_k\}$ s.t.

$$\mathbb{P}\left[\sup_{0 \le t \le T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k}\right] < 2^{-k}$$

Now, we have a fast diminishing probability so by Borell-Candelli Lemma:

$$\mathbb{P}\left[\sup_{0 \le t \le T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k}, \text{ infinitely often}\right] = 0$$

$\Rightarrow$ for a.a. ω, $\exists$ $K_1(\omega)$ s.t.

$$\sup_{0\le t\le T} \left| I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega) \right| \le 2^{-k} \quad \forall k \ge K_1(\omega)$$

$\Rightarrow$ $I_{n_k}(t,\omega)$ is uniformly convergent for $t\in[0,T]$, f. a.a. ω.

$\to$ denote the limit by $J(t,\omega)$ which is continuous in t, a.s.!

since $I_{n_k}(t,\cdot) \longrightarrow I(t,\cdot)$ in $\mathcal{L}^2(\mathbb{P})$ $\forall t$, we must have $\quad I(t,\cdot) = J(t,\cdot)$ a.s., $\forall t\in[0,T]$.

So, we conclude that $I(t,\omega)$ is continuous in t, a.s.

The last argument: as $I_n \to I$ in $\mathcal{L}^2(\mathbb{P},\mu)$, by property (4)

$$\begin{aligned}
\mathbb{E}\left[I_t \mid \mathcal{F}_s^W \right] &= \mathbb{E}\left[\mathcal{L}^2\text{-}\lim_{n\to\infty} I_n(t) \,\middle|\, \mathcal{F}_s^W \right] \\
&= \mathcal{L}^2\text{-}\lim_{n\to\infty} \mathbb{E}\left[I_n(t) \mid \mathcal{F}_s^W \right] \\
&= \mathcal{L}^2\text{-}\lim_{n\to\infty} I_n(s) = I_s. \quad \Rightarrow I_t \text{ is martingale}
\end{aligned}$$

Finally, by Doob's martingal Inequality:

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le t\le T} |I_t| > \lambda \right) &\le \frac{1}{\lambda^2} \mathbb{E}\left[|I_T|^2 \right] \\
&= \frac{1}{\lambda^2} \mathbb{E}\left[\left(\int_0^T f\, dW_t \right)^2 \right] \\
\text{by Itô isometry} \quad &= \frac{1}{\lambda^2} \mathbb{E}\left[\int_0^T f^2\, dt \right] \quad \blacksquare
\end{aligned}$$