Lecture 6

Wednesday, January 21, 2026 1:42 PM

Last time, we saw

1) Wiener Process for real:
   the "unique" <u>continuous-path</u> process with <u>independent increaments</u> <u>distributed</u> as $W_t - W_s \sim \mathcal{N}(0, t-s)$, $\forall t > s$.

2) the natural filteration: $\mathcal{F}_t^W \triangleq \sigma(\{W_s : 0 \le s \le t\})$.

3) Properties of W_t:
   → It "exists uniquely" (Donsker's theorem):

$$X_t(N) = \sum_{k=1}^{\lfloor Nt \rfloor} \frac{\xi_i}{\sqrt{N}}, \quad \{\xi_i\} \text{ i.i.d zero-mean, unit variance}, \ t \in [0,1].$$

$X_\bullet(N) \to W_\bullet$ "in Law" meanig that

$\mathbb{E}(f(X_\cdot(N))) \to \mathbb{E}(f(W_\cdot))$ bounded, continuous functionals f on $C([0,1])$!

→ no-where differentiable: $\frac{\partial^2}{\partial \tau \partial t} \mathbb{E}[W_t W_\tau] = \delta(t-\tau)$, δ- Dirac func.

→ is a $\mathcal{F}_t^W$- wiener process: $W_t \in \mathcal{F}_t^W$, and $W_t - W_s \perp \mathcal{F}_s^W$.

→ is a $\mathcal{F}_t^W$- martingal: $W_t \in \mathcal{F}_t^W$, and $\mathbb{E}[W_t | \mathcal{F}_s^W] = W_s$, $\forall t \ge s$.

→ is a $\mathcal{F}_t^W$- Markov process:

$$\mathbb{E}[f(W_t) \mid \mathcal{F}_s^W] = \mathbb{E}[f(W_t) | W_s], \quad \forall t > s, \ \forall f \text{ bounded and measurable}$$

★ Note: If you wanna answer any question about W_t, it will be very useful to connect it to its increments $W_t - W_s$!

<u>Exercise</u>: show any process with independent increment is a markov process. More precisely, consider $\{X_t : t \ge 0\}$ and $\mathcal{F}_t \triangleq \sigma(\{X_s : 0 \le s \le t\})$. If $X_t - X_s \perp \mathcal{F}_s$, then X_t is a $\mathcal{F}_t$-markov process.

process with <u>independent</u>, <u>centered</u> increments

Exercise: Also, show that any ... - ...

is a martingale.

Exercise: show that these are martingales: 1) $W_t^2 - t$, 2) $\exp(\theta W_t - \theta^2 t/2)$
θ fixed.

The Itô Integral: $\{W_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$

Remember our goal is to make sense of the general SDE

$$\text{“} dX_t = f(X_t, t)\, dt + \sigma(X_t, t)\, dW_t \text{”}$$

as $$X_t = \int_0^t f(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dW_s$$

so we need to define "$\int_0^T \sigma(X_t, t)\, dW_t$" !

Lets start slow and intuitively: $(T < \infty)$

1- suppose $\sigma(\omega, t) = \sigma$ is constant, then

$$\int_0^T \sigma\, dW_t := \sigma(W_T - W_0) = \sigma W_T$$

2- suppose $\sigma(\omega, t) = \sigma_t$ is a piece-wise constant on each $[t_i, t_{i+1}]$,

then $$\int_0^T \sigma_t\, dW_t = \sum_{t_i} \sigma(t_i)\,[W_{t_{i+1}} - W_{t_i}]$$

3- suppose $\sigma(\omega, t) = \sigma(t)$ be any $\mathcal{B}([0,T])$-measurable function, then

$$I(\sigma) \triangleq \int_0^T \sigma(t)\, dW_t := \lim_{n\to\infty} I(\sigma_n) \triangleq \sum_{t_i \in \Pi_n} \sigma_{\Pi_n}(t_i)\,[W_{t_{i+1}} - W_{t_i}]$$

where Π_n is an increasingly finer seq. of partitions of $[0,T]$, i.e.,

$\Pi_n : 0 \leq t_1 < \dots < t_n = T$ and $\sup_{t_i \in \Pi_n} |t_{i+1} - t_i| \to 0$ as $n \to \infty$.

and σ_{Π_n} is a seq. of simple functions converging to σ:

$$\sigma_{\Pi_n}(t) = \sum_{t_i \in \Pi_n} \sigma(t_i)\, \chi_{[t_i, t_{i+1})}(t) \longrightarrow \sigma(t) \text{ (uniformly in } t)$$

... $\mathcal{B}([0,T])$-measurable function, then

4 - Suppose $\sigma(w,t)$ is any $\mathcal{F}\times\mathcal{B}([0,T])$-m...

$I(\sigma)$ is defined similarly, <u>w-by-w</u>; more specifically,

$$I(\sigma)(w) = \lim_{n\to\infty} \sum_{t_i\in\pi_n} \sigma_{\pi_n}(w,t_i)\left[W_{t_{i+1}}(w) - W_{t_i}(w)\right]$$

where $\sigma_{\pi_n}(w,t) = \sum_{t_i\in\pi_n} \sigma(w,t_i)\,\chi_{[t_i,t_{i+1})}(t)$ called <u>an elementary seq.</u>

<u>Question</u>: does this converge $I(\sigma_n) \to I(\sigma)$?! in what sense!

1) The convergence in usual (Lebesgue-Stieltjes) sense <u>cannot happen!</u>
because showing that $I(\sigma_n) \to I(\sigma)$ uniformly is equivalent to showing that $I(\sigma_n)$ is a Cauchy seq:

$$\left|I(\sigma_n) - I(\sigma_m)\right| = \left|\sum_i \left[\sigma_{\pi_n}(t_i) - \sigma_{\pi_m}(t_i)\right]\left[W_{t_{i+1}} - W_{t_i}\right]\right|$$

$$\leq \sup_{t\in[0,T]} \left|\sigma_{\pi_n}(t) - \sigma_{\pi_m}(t)\right| \cdot \underbrace{\sup_{\pi\in\Pi_{(0,T)}} \sum_i \left|W_{t_{i+1}} - W_{t_i}\right|}_{\text{"total variation"} := TV(W_\cdot,0,T)}$$

$\Pi_{(0,T)}$: set of finite partions of $(0,T)$.

$$= TV(W_\cdot, 0, T) \cdot \left\|\sigma_{\pi_n} - \sigma_{\pi_m}\right\|_\infty$$

Now, if $TV(W_\cdot,0,T) < \infty$ then $\sigma_{\pi_n} \to \sigma$ uniformly implies that $\{I(\sigma_n)\}$ is a Cauchy sequence of $\mathbb{R}$ and that $I(\sigma_n) \to I(\sigma)$.

But Lemma: $TV(W_\cdot, 0, T) = \infty$ with prob. 1.

$\Rightarrow$ This means that $W_\cdot$ has <u>infinite variations</u> and thus $I_n(\sigma)$ <u>may NOT converge</u> even if σ_{π_n} converges uniformly!

<u>A Bad but Insightful Example</u>:

Define $h_n(t_i) = \text{sign}(W_{t_{i+1}} - W_{t_i})$ for $t_i \in \pi_n$ where for seq. $\{\pi_n\}$: $\sum |W_{t_{i+1}} - W_{t_i}| \to \infty$ as $n\to\infty$.

the seq. $\{h_n\}$ ... $\sum_{t_i \in \Pi_n}$ ...

Ex: show that $I(h_n) \to \infty$!!

Now, define the seq. of simple functions $\sigma_n \triangleq [I(h_n)]^{-1/2} \cdot h_n$

Then, $I(\sigma_n) = [I(h_n)]^{1/2} \to \infty$ as $I(h_n) \to \infty$

but $\sigma_n \to 0$ uniformly (bc $\sup_t |\sigma_n(t)| = I(h_n)^{-1/2}$) !!

<u>Conclusion:</u> This bad example in defining $h_n(t_i)$ uses

(I) <u>future information at time t_i</u>, and

(II) <u>looks for convergence uniformly in t.</u>

<u>Resolving these issues for defining $I(\sigma)$:</u>

(I) $\to$ The first one is easy; we just require

$$\boxed{\sigma_t \in \mathcal{F}_t, \ \forall t}$$

(II) $\to$ For the second, we look for $\boxed{\text{Convergence in } \mathcal{L}_2}$ because W_t has finite Quadratic variance on $[0,T]$; i.e.

<u>Lemma 1:</u> W_t has finite Quadratic variance on $[0,T]$:

$$\sum_{t_i \in \Pi_n} (W_{t_{i+1}} - W_{t_i})^2 \xrightarrow{n \to \infty} T \quad \text{in } \mathcal{L}^2$$

<u>Proof:</u> $\mathbb{E} \sum_{t_i \in \Pi_n} (W_{t_{i+1}} - W_{t_i})^2 = \sum_{t_i \in \Pi_n} (t_{i+1} - t_i) = T - 0 = T$.

now, define the centered r.v. $Z_i = (W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)$ for $t_i \in \Pi_n$

so that $Z_i \perp Z_j$, $i \neq j$. and $Z_i \overset{\text{in law}}{=} (\xi^2 - 1)(t_{i+1} - t_i)$, $\xi \sim \mathcal{N}(0,1)$

So that ...

Now,
$$\mathbb{E}\left[\left(\sum_{t_i}\left(W_{t_{i+1}}-W_{t_i}\right)^2-T\right)^2\right]=\mathbb{E}\left[\sum_{t_i\in\Pi_n}(Z_i)^2\right]$$
$$=\mathbb{E}\left[\left(\xi^2-1\right)^2\right]\sum_{t_i\in\Pi_n}\left(t_{i+1}-t_i\right)^2$$
$$=\mathbb{E}\left[(\xi^2-1)^2\right]\cdot T\cdot\sup_{t_i\in\Pi_n}\left|t_{i+1}-t_i\right|^2\longrightarrow 0 \quad \text{as } n\to\infty$$

for any increasingly finer seq of partitions $\{\Pi_n\}$. ∎

Lemma 2: If $\varphi(\omega,t)$ is bounded and elementary; i.e.
$$\varphi(\omega,t)=\sum_{i\in\Pi_n}e_i(\omega)\cdot\chi_{[t_i,t_{i+1})}(t), \text{ then}$$

Itô isometry

$$\mathbb{E}\left[\left(\int_0^T\varphi_t\,dW_t\right)^2\right]=\mathbb{E}\left[\int_0^T\varphi_t^2\,dt\right]$$

Proof: Set $\Delta W_j=W_{t_{i+1}}-W_{t_i}$, and compute
$$\mathbb{E}\left[e_ie_j\,\Delta W_i\,\Delta W_j\right]=\begin{cases}\mathbb{E}\left[e_i^2\right]\cdot(t_{i+1}-t_i) & \text{if } i=j\\ 0 \rightsquigarrow \Delta W_i\perp\Delta W_j & \text{o.w.}\end{cases}$$

So
$$LHS=\sum_{i,j}\mathbb{E}\left[e_ie_j\cdot\Delta W_i\cdot\Delta W_j\right]=\sum_i\mathbb{E}\left[e_i^2\right](t_{i+1}-t_i)=RHS.$$ ∎