Lecture 5

Wednesday, January 14, 2026 9:36 AM

- wiener Process for real
- its properties

Def: $\{W_t, t \geq 0\}$ is called a standard wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ if

1) W_t has continuous sample paths: $t \mapsto W_t(\omega)$ is continuous for almost all $\omega \in \Omega$; and

2) choose any finitely many $0 \leq t_0 < t_1 < \cdots < t_n$, $(n < \infty)$ and consider $(W_{t_0}, W_{t_1}, \ldots, W_{t_n})$ as a vector of random variables. then, $\{W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}\}$ is a collection of independent Gaussian r.v.s with zero-mean and variances $\{t_0, t_1 - t_0, \ldots, t_n - t_{n-1}\}$, respectively.

Def: n-dim wiener process $(W_t^1, \ldots, W_t^n)$, $t \geq 0$ where $(W_t^i, i = 1, 2, \ldots)$ are independent standard wiener process.

Def: Given a filtration $\{\mathcal{F}_t\}$, we say W_t is a $\mathcal{F}_t$-wiener process if

1) W_t is $\mathcal{F}_t$-adapted, and 2) $W_t - W_s$ is independent of $\mathcal{F}_s$ $\forall t > s$.

we often work with the natural filteration $\mathcal{F}_t^W \triangleq \sigma(\{W_s : 0 \leq s \leq t\})$.

Properties of wiener process:

1) W_t is a Gaussian r.v. for each $t \geq 0$
2) $\mathbb{E}[W_t] = 0$, $\mathbb{E}[W_t W_s] = \min(t, s)$

$\Big\}$ $W_t \sim \mathcal{N}(0, t)$ $\forall t \geq 0$

3) $W_0 = 0$ a.s.

4) W_t is a $\mathcal{F}_t^W$ - wiener process

5) Consider the filtration $\{\mathcal{F}_t^W\}$ as $\mathcal{F}_t^W = \sigma(W_s : s \leq t)$. Then

W_t is a "$\mathcal{F}_t^W$ - martingale", i.e,

I) W_t is $\mathcal{F}_t^W$ - adapted and II) $\mathbb{E}[W_t | \mathcal{F}_s^W] = W_s$, $\forall t > s$

6) W_t is a $\mathcal{F}_t^W$ - markov process; i.e.,

$\mathbb{E}[f(W_t) | \mathcal{F}_s^W] = \mathbb{E}[f(W_t) | W_s]$, $\forall t > s$, $\forall f$ bounded and measurable

7) W_t is nowhere differentiable (a.s.).

8) W_t is unique in the following sense:

if W_t, W_t' are wiener processes, the $C([0,\infty))$ - valued r.v.s

$W_\cdot, W_\cdot' : \Omega \longmapsto (C([0,\infty)), \mathcal{C})$ mapping ω to the path $W_\cdot(\omega)$

have the same law (, where $\mathcal{C} = \sigma(\Pi_t : t \in [0,\infty))$ and

$\Pi_t : x \in C([0,\infty)) \longmapsto x_t$ is the evaluation mapping).

9) W_t, $t \geq 0$ exist!

Proof: 1) is trivial, for 2) note that $\mathbb{E}[W_t W_s] = \mathbb{E}[W_t^2] - \mathbb{E}[W_t(W_s - W_t)]$.

and for 3) $W_0 \in \mathcal{N}(0,0) \Rightarrow \mathbb{P}(W_0 = 0) = 1$. For 4) Consider increments

$W_t - W_s$ and $W_s - W_0$ which are independent and note $\sigma(W_s) = \sigma(W_s - W_0)$.

For 5) note $W_s \in \mathcal{F}_s^W$ and $W_t - W_s \perp \mathcal{F}_s$ (by def.)

$$\Rightarrow \mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$$

For 6) we need to show that

$$\mathbb{E}[f(W_t) | W_s] = g(W_s) = \mathbb{E}[f(W_t) | \mathcal{F}_s]$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined as:

$$g(z) \triangleq \mathbb{E}\left[f(W_t - W_s + z) \mid \mathcal{F}_s\right] \underset{\uparrow}{=} \mathbb{E}\left[f(W_t - W_s + z)\right] = \mathbb{E}_X\left[f(X+z)\right]$$

because $W_t - W_s \perp \mathcal{F}_s$

where the random variable $X \triangleq W_t - W_s$ has the law $N(0, t-s)$ and $X \perp W_s$.

suppose the RHS is true, then the LHS follows by

$$\mathbb{E}\left[f(W_t) \mid W_s\right] \underset{\text{Tower prop. } W_s \in \mathcal{F}_s.}{=} \mathbb{E}\left[\mathbb{E}\left[f(W_t) \mid \mathcal{F}_s\right] \mid W_s\right] \underset{\text{RHS}}{=} \mathbb{E}\left[g(W_s) \mid W_s\right] \underset{g(W_s) \in \sigma(W_s).}{=} g(W_s)$$

Now, it is left to show RHS: we wanna use kolmogorov def. to show:

$$\mathbb{E}\left[f(W_t) 1_A\right] = \mathbb{E}\left[g(W_s) 1_A\right] \text{ for all } A \in \mathcal{F}_s.$$

Define r.v.s $X \triangleq W_t - W_s$ and $Y \triangleq (W_s, 1_A) \Rightarrow X \perp Y$ (why?)

$\Rightarrow$ the Law of (X, Y) is $\mu_X \times \mu_Y$ and by Fubini's Thm. (f is bounded)

$$\begin{aligned}
\mathbb{E}\left[f(W_t) 1_A\right] &= \int f(x+w)\, a\, \mu_X(dx) \cdot \mu_Y(dw, da) \\
&= \int \left[\int f(x+w)\, \mu_X(dx)\right] a\, \mu_Y(dw, da) \\
&= \int g(w)\, a\, \mu_Y(dw, da) = \mathbb{E}\left[g(W_s) 1_A\right].
\end{aligned}$$

Note: Here is an equivalent way of thinking about the last part

$$\begin{aligned}
\mathbb{E}\left[f(W_t) 1_A\right] &= \mathbb{E}_{X,Y}\left[f(X+Y_1)\, Y_2\right] \\
&= \mathbb{E}_Y\left[\mathbb{E}_X\left[f(X+Y_1)\right] \cdot Y_2\right] && \text{because } X \perp Y \\
&= \mathbb{E}_Y\left[g(Y_1) \cdot Y_2\right] && \text{by definition of } g \\
&= \mathbb{E}\left[g(W_s) \cdot 1_A\right] && \text{by def. of } Y.
\end{aligned}$$

for 7) suppose $W \in \mathcal{L}^2$, then $R_{WW}(t,\tau) = \mathbb{E}[W_t W_\tau] < \infty$. We can show that [Thm 5.14 in Speyer-Chung] X_t is "mean-squared" differentiable if and only if $\frac{\partial^2}{\partial t \partial \tau} R_{XX}(t,\tau)$ exists at (t,t)! But for W_t

(see lectures)

$$R_{WW}(t,\tau) = \min(t,\tau) = \begin{cases} t & t<\tau \\ \tau & t \geq \tau \end{cases}$$

$$\Rightarrow \frac{\partial}{\partial t} R_{WW}(t,\tau) = \begin{cases} 1 & t<\tau \\ 0 & t>\tau \end{cases} \Rightarrow \frac{\partial}{\partial \tau \partial t} R_{WW}(t,\tau) = \text{"}\delta(t-\tau)\text{"}$$

but this means that $\frac{\partial}{\partial \tau \partial t} R_{WW}(t,\tau)$ doesn't exist at any (t,t).

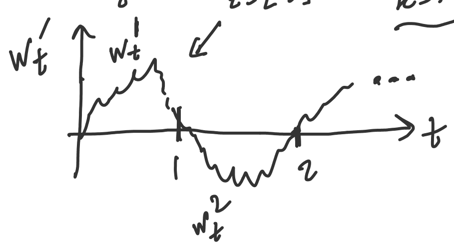
for 8) the proof uses technology that will not be useful for us!

for 9) suppose $\{W_t : t \in [0,1]\}$ exists on some $(\Omega, \mathcal{F}, \mathbb{P})$. then we can do the following useful construction to extend it on $[0,\infty)$.

1) Get i.i.d copies of $\{W_t^n : t \in [0,1]\}$ for $n = 1, 2, \ldots$

2) Define $\Omega' = \Omega \times \Omega \times \cdots$, $\mathcal{F}' = \mathcal{F} \times \mathcal{F} \times \cdots$, $\mathbb{P}' = \mathbb{P} \times \mathbb{P} \times \cdots$

3) Define $$W_t' = W_{t - \lfloor t \rfloor}^{\lfloor t \rfloor + 1} + \underbrace{\sum_{k=1}^{\lfloor t \rfloor} W_1^k}$$ for $t \in [0,\infty)$.



to keep continuity we start W_t^2 where W_t^1 is finished.

why $\{W_t : t \in [0,1]\}$ exists? "you have seen the lectures"

Recall $X_t(N) = \sum_{i=1}^{\lfloor Nt \rfloor} \frac{\xi_i}{\sqrt{N}}$ where $\{\xi_i\}$ are from any distribution

but i.i.d zero mean, unit variance.

Thm (Donsker): $X_t(N) \to W_t$ "in law" as $N \to \infty$

meaning that $\mathbb{E}(f(X_\cdot(N))) \to \mathbb{E}(f(W_\cdot))$

for all bounded, continuous functionals f on $C([0,1])$!

proof: uses Skorokhods topology and embedding thm! ∎

Discussion on Donsker's Thm:

Recall that by Riesz Representation theorem, each bounded linear functional $f(\cdot)$ on $C[0,1]$ (cont. func. on $[0,1]$)

is essentially an integration against a finite signed measure

on $[0,1]$; so

$$\mathbb{E}(f(X_\cdot(N))) = \mathbb{E}\left[\int_0^1 X_s(N)\, \mu_f(ds)\right] \quad \text{for some } \mu_f$$

$$= \int_0^1 \mathbb{E}[X_s(N)]\, \mu_f(ds) \quad \text{by Fubinni}$$

$$= 0, \quad \forall N.$$

So, the usefulness of Donsker's theorem is about any bounded continuous nonlinear functional on $C[0,1]$. For example,

Choose $f: C[0,1] \to \mathbb{R}$ to be

$$f(X_t) = \sup_t X_t \quad \text{for any } X_\cdot \in C[0,1]$$

then $\mathbb{E}[\sup_t W_t : t \in [0,T]] = \lim_{N\to\infty} \mathbb{E}[\sup_t X_t(N) : t \in [0,T]]$