Lecture 2

Monday, January 5, 2026 7:39 PM

→ In this Lecture, we review some of the basic concepts in order to study Wiener process.

Consider a set Ω of all possible outcomes.

Definition 1.1.5. A *σ-algebra* $\mathcal{F}$ is a collection of subsets of Ω such that

1. If $A_n \in \mathcal{F}$ for countable n, then $\bigcup_n A_n \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. $\Omega \in \mathcal{F}$.

$\Rightarrow$ Countable intersection ✓

$\Rightarrow \phi \in \mathcal{F}$.

An element $A \in \mathcal{F}$ is called an *($\mathcal{F}$-)measurable set* or an *event*.

Definition 1.1.7. Let $\{A_i\}$ be a (not necessarily countable) collection of subsets of Ω. Then $\mathcal{F} = \sigma\{A_i\}$ denotes the smallest σ-algebra that contains every set A_i, and is called the σ-algebra *generated by* $\{A_i\}$.

It is obvious that there exist at least one σ-algebra that contains every set A_i; namely the power set.
To show uniqueness, note that if $\{F_j\}$ is a collection of σ-algebras (not nec. countable) then $\bigcap_j F_j$ is a σ-algebra.

– How to think of a σ-algebra F?
the collection of information that is available to us!

Ex: $\Omega = \{1, 2, \ldots, 6\}$, $F = \sigma(\{1\}, \{6\})$.
then we can answer questions like:
- Did 1 happen? did 6 happen?
- Did any thing but 1 or 6 happened? $\to \{1,6\}^c$

...

Definition 1.1.11. A *probability measure* is a map $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

1. For countable $\{A_n\}$ s.t. $A_n \cap A_m = \varnothing$ for $n \neq m$, $\mathbb{P}(\bigcup_n A_n) = \sum_n \mathbb{P}(A_n)$.
2. $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$.

"Countable additivity of disjoint events"

Def: A probability space is a triple $(\Omega, F, \mathbb{P})$.

Example: Consider Borel σ-algebra of $\mathbb{R}$; i.e, the

σ-algebra generated by open sets of $\mathbb{R}$,
denoted by $\mathcal{B}(\mathbb{R})$.

Theorem 1.1.15. *Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the function $F(x) = \mathbb{P}(]-\infty, x])$ is nondecreasing, right-continuous, and $F(x) \to 0$ as $x \to -\infty$, $F(x) \to 1$ as $x \to \infty$. Conversely, for any function $F(x)$ with these properties, there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(x) = \mathbb{P}(]-\infty, x])$.*

- F is the CDF for $\mathbb{P}$ on $\mathbb{R}$.
- if $\mathcal{F}$ is the power set of $\mathbb{R} \Rightarrow \nexists$ any probability meas. on $\mathcal{F}$.

Lemma: (We can take limits of monotone sequences!)

Consider $(\Omega, \mathcal{F}, \mathbb{P})$

- $A_1 \subset A_2 \subset \cdots \in \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_n A_n)$
- $A_1 \supset A_2 \supset \cdots \in \mathcal{F} \Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcap_n A_n)$

proof: By using Countable Additivity of disjoint events:

define disjoint events $B_1 = A_1$, $B_k = A_k \setminus A_{k-1}$ ...

$\Rightarrow \mathbb{P}(B_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}) \Rightarrow$

$$\mathbb{P}(\bigcup_n A_n) = \mathbb{P}(\bigcup_n B_n) = \mathbb{P}(A_1) + \sum_{k=2}^{\infty}[\mathbb{P}(A_k) - \mathbb{P}(A_{k-1})]$$
$$= \lim_{n\to\infty} \mathbb{P}(A_n). \quad \blacksquare$$

Infinitly often event: $\{A_n\}_1^\infty \subset \mathcal{F}$

$\omega \in A_n \iff \omega \in \bigcup_{k\geq n} A_n, \forall n \iff \omega \in \bigcap_{n\geq 1} \bigcup_{k\geq n} A_n \triangleq \limsup_{n\to\infty} A_n$

Lemma (Borell-Cantelli):

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0.$$

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measurable functions $\longleftrightarrow$ Random variables:

Definition 1.3.1. An *($\mathcal{F}$-)measurable function* is a map $f : \Omega \to S$ from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$ such that $f^{-1}(A) \equiv \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$ for every $A \in \mathcal{S}$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(S, \mathcal{S})$ is a measurable space, then a measurable function $f : \Omega \to S$ is called an *S-valued random variable*. A real-valued random variable $(S = \mathbb{R}, \mathcal{S} = \mathcal{B}(\mathbb{R}))$ is often just called a *random variable*.

Notation: Consider R.V. $X : \Omega \to S$, $(\Omega, \mathcal{F}, \mathbb{P})$, $A \in \mathcal{S}$.

$$(X \in A) = \{\omega \in \Omega ; X(\omega) \in A\} = X^{-1}(A)$$

$$\Rightarrow \mathbb{P}(X \in A) = \mathbb{P}(\{\omega \in \Omega ; X(\omega) \in A\}) .$$

- σ-algebra generated by a function :

  Consider a set Ω, a meas. space $(S, \mathcal{S})$, and $h : \Omega \to S$. Then, the σ-algebra generated by h is $\sigma(h) = \sigma\{h^{-1}(A) : A \in \mathcal{S}\}$ and h will be measurable w.r.t. $(\Omega, \sigma(h))$.

- What is the meaning of $\sigma(X)$ for a R.v. $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$?

  you should view it as the set of information that allows us to answer those yes-no question by <u>measuring the X</u> !

  example : - Toss a fair coin twice :

  $$\to \Omega = \{HH, HT, TH, TT\} , \mathcal{F} = 2^{\Omega} .$$

  - suppose we only observe the outcome of the first coin : $\to$ $X : \Omega \to \{0, 1\}$ defined as

  $$X(HH) = X(HT) = 1 , \quad X(TH) = X(TT) = 0 .$$

  - what is $\sigma(X)$?

  $$\sigma(X) = \{\phi, \underbrace{\{HH, HT\}}_{\uparrow}, \{TH, TT\}, \Omega\} \in \mathcal{F}$$

  this event say the first coin come up heads !

For $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$E[X] = \int X \, d\mathbb{P} = \int X(\omega) \, \mathbb{P}(d\omega)$$

$\nearrow$ Rieman integral (if all functions are cont.)

$\searrow$ "Lebesgue integral" (if functions are only measurable)

$\to$ Ex: Review Lebesgue interal defined as the limit of integrals constructed by so called seq of "simple functions" $\{X_n\}$ converging "monotonically" to X : $X_n \nearrow X$.

Convergingly ...

→ Ex: Review basic properties of $E(x)$ like:

- $|\mathbb{E}(x)| \le \mathbb{E}(|x|)$
- if $|\mathbb{E}(x)| < \infty \Rightarrow |X| < \infty$, a.s.
- if $x \geq 0$ a.s. and $\mathbb{E}(x) = 0 \Rightarrow X = 0$ a.s.
- Chebyshev/Markov inequality:
$$\mathbb{P}(|x| > \alpha) \le \frac{\mathbb{E}(|x|)}{\alpha}$$
- Jensen inequality: $g(\cdot)$ is real-valued convex func.
$$|\mathbb{E}(x)| < \infty \Rightarrow \mathbb{E}[g(x)] \geq g(\mathbb{E}(X)).$$

The $\mathcal{L}^p$ space: $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) = \{X: \Omega \to \mathbb{R} : \|X\|_p < \infty\}$

where $\|x\|_p \triangleq [E(|x|^p)]^{1/p}$ for $p \geq 1$.

- $\mathcal{L}^p \subset \mathcal{L}^q$ if $p \geq q$.
- $\mathcal{L}^p$ is linear: $\alpha X + \beta Y \in \mathcal{L}^p$ if $X, Y \in \mathcal{L}^p$, $\alpha, \beta \in \mathbb{R}$.
- Hölder ineq: $|\langle X, Y\rangle| \triangleq |E[XY]| \le \|X\|_p \cdot \|Y\|_q$
where $p^{-1} + q^{-1} = 1$; i.e.
if $X \in \mathcal{L}^p$, $Y \in \mathcal{L}^q \Rightarrow XY$
- Minkowski ineq: $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$.

Modes of Convergence:

Definition 1.5.1. Let X be a random variable and $\{X_n\}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $X_n \to X$ *a.s.* if $\mathbb{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$ (why is this event in $\mathcal{F}$?).
2. $X_n \to X$ *in probability* if $\mathbb{P}(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$.
3. $X_n \to X$ *in $\mathcal{L}^p$* if $\|X_n - X\|_p \to 0$ as $n \to \infty$.
4. $X_n \to X$ *in law* (or *in distribution*, or *weakly*) if $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X))$ for every bounded continuous function f.

Proposition 1.5.2. *Let X be a random variable and $\{X_n\}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following implications hold:*

1. $X_n \to X$ *a.s.* $\implies$ $X_n \to X$ *in probability.* $\not\Leftarrow$
2. $X_n \to X$ *in* $\mathcal{L}^p$ $\implies$ $X_n \to X$ *in probability.* $\not\Leftarrow$
3. $X_n \to X$ *in* $\mathcal{L}^p$ $\implies$ $X_n \to X$ *in* $\mathcal{L}^q$ $(q \le p)$. $\not\Leftarrow$
4. $X_n \to X$ *in probability* $\implies$ $X_n \to X$ *in law.* $\not\Leftarrow$

} in general!

Proof:

1) $\limsup \mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{P}(\limsup \{|X_n - X| > \varepsilon\})$
$= \mathbb{P}(|X_n - X| > \varepsilon, \text{ i.o}) = 0.$

2) By chebyshev's ineq. $\mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{E}[|X_n - X|^p]\varepsilon^{-p} \to 0.$

3) $x \mapsto x^{p/q}$ is convex, so by Jensen's inequality:
$$(\mathbb{E}[|X_n - X|^q])^{p/q} \le \mathbb{E}[|X_n - X|^p] \to 0.$$

4) $|\mathbb{E}[f(X_n) - f(X)]| \le \mathbb{E}|f(X_n) - f(X)|$
$\le \mathbb{E}[\mathbb{1}\{|f(X_n) - f(X)| > \varepsilon\} |f(X_n) - f(X)|]$
$+ \mathbb{E}[\mathbb{1}\{|f(X_n) - f(X)| < \varepsilon\} \cdot |f(X_n) - f(X)|]$

(by cont. of f, $\exists\, \delta > 0$) $\Rightarrow$ $\le 2\|f\|_\infty \cdot \underbrace{\mathbb{P}(|X_n - X| > \delta)}_{\to 0 \text{ as } n \to \infty} + \varepsilon$, $\forall \varepsilon > 0.$

so $\limsup |\mathbb{E} f(X_n) - \mathbb{E} f(X)| \le \varepsilon$, $\forall \varepsilon \Rightarrow X_n \to X$ in law $\blacksquare$

Important Non-examples:

(E.1) convergence in law $\not\Rightarrow$ convergence in probability

$\Omega = \{a_1, a_2\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}(\{a_1\} = 1/2) \to$ fair coin

$X: \Omega \to \{\pm 1\}$: $X(a_1) = -X(a_2) = 1$.

$X_n \triangleq (-1)^n X$.

Then $X_n \to X$ in law (distribution) but not in any other form! in particular, the sample pathes $n \mapsto X_n(w)$ for a fixed w is just fluctuating!

very weak convergence!

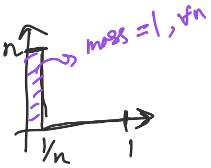
why?!

$\mathbb{E}(f(X_n)) = \mathbb{E}(f((-1)^n X)) = \int f((-1)^n X(w))\, \mathbb{P}(dw)$

$$= f((-1)^n)\cdot \tfrac{1}{2} + f((-1)^n(-1))\cdot \tfrac{1}{2}$$
$$= \tfrac{1}{2}\left(f((-1)^n) + f((-1)^{n+1}) \right)$$
$$\overset{\text{why}}{=} \tfrac{1}{2}\left(f(1) + f(-1) \right)$$

Consider $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P}$ is uniform meas. on $[0,1]$.

(E.2) Convergence a.s. but not in $\mathcal{L}^p$:



$$X_n(\omega) = n\, I_{(0,\frac{1}{n}]}(\omega), \quad \omega \in [0,1].$$

$X_n \to 0$ a.s. but $\|X_n\|_1 = 1 \Rightarrow X_n \not\to 0$ in $\mathcal{L}^1$

$\Rightarrow X_n \not\to 0$ in any $\mathcal{L}^p$, $p \geq 1$.

(E.3) $(q<p)$ Convergence in $\mathcal{L}^q$ but not in $\mathcal{L}^p$:

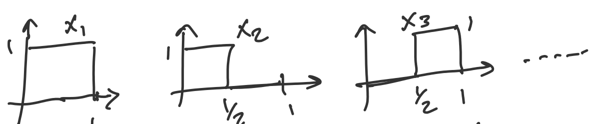
$$X_n(\omega) = n^{1/p}\, I_{(0,1/n]}(\omega).$$

$X_n \to 0$ in $\mathcal{L}^q$, $q<p$ but not for any $q \geq p$!

(E.4) Convergence in $\mathcal{L}^p$ but not a.s.: "Dancing sequence" or "Typewriter seq."

$\{X_n\} \triangleq$
- For each $n = 1,2,3,\ldots$
- → devide $[0,1]$ to 2^{n-1} equal intervals, U_k^n, $k = 1,2,\ldots,2^{n-1}$
- → sequentially from left to right, set

$$X_{n+k}(\omega) = I_{U_k^n}(\omega) = \begin{cases} +1 & \text{if } \omega \in U_k \\ 0 & \text{o.w.} \end{cases}$$



$X_n \to 0$ in $\mathcal{L}^p$ for any $p \geq 1$. $\left(E(|X_n|^p) = |U_k^n| \to 0 \text{ as } n \to \infty \right)$

But, fix any $\omega \in [0,1]$, the sequence $X_n(\omega)$ always fultuates between $0,1$ $\Rightarrow X_n(\omega) \not\to 0$ for any $\omega \in [0,1]$.

$\Rightarrow X_n(\omega) \not\to 0$ a.s.

In (E.4) we learned $X_n \to X$ in prob. $\not\Rightarrow X_n \to X$, a.s..

because possible irregularity like the "dancing seq."

<u>Question</u>: Is there an addition regularity assumption s.t. convergence in prob $\Rightarrow$ conv. a.s. ?!

<u>Lemma</u>: if $\underbrace{x_n \to x \text{ in prob.}}_{\mathbb{P}(|x_n - x| > \varepsilon) \to 0,\ \forall \varepsilon > 0}$ and $\sum_n \mathbb{P}(\underbrace{|x_n - x| > \varepsilon}_{A_n}) < \infty$, $\forall \varepsilon > 0$. $\Rightarrow x_n \to x$ a.s.

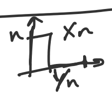
<u>Proof</u>: By Borell–Cantelli, $\mathbb{P}(\overbrace{|x_n - x| > \varepsilon, \text{ i.o.}}^{\limsup A_n}) = 0$, $\forall \varepsilon > 0$.

Thus, $\mathbb{P}(\text{fix any } k, |x_n - x| > 1/k \text{ for finite \# of } n\text{'s}) = 1$.

So, for any fixed k, $\exists N$ s.t. $\forall n \geq N$ $|x_n(\omega) - x(\omega)| \leq 1/k$ for a.e. $\omega \in \Omega$.

<u>Corollary</u>: if $x_n \to x$ in prob., then there exist a subsequence $x_{n_k} \to x$ a.s..

Proof: $\mathbb{P}(|x_n - x| > \varepsilon) \to 0$; choose the subsequence $n_k \uparrow \infty$ such that $\mathbb{P}(|x_{n_k} - x| > \varepsilon) \leq \frac{1}{2^k}$. Apply Lemma above. $\blacksquare$

In (E.2), we learned that (by ) $X_n \to X$, a.s. $\not\Rightarrow$ $x_n \to x$ in $\mathcal{L}^p$.

this also automatically means $x_n \to x$ in prob $\not\Rightarrow$ $x_n \to x$ in $\mathcal{L}^p$.

Question: Is there an additional regularity that allows this ?!

(DCT) **Theorem 1.5.12 (Dominated convergence).** *Let $X_n \to X$ in probability, and suppose there exists a nonnegative $Y \in \mathcal{L}^p$, with $p \geq 1$, such that $|X_n| \leq Y$ a.s. for all n. Then X and X_n are in $\mathcal{L}^p$ for all n, $X_n \to X$ in $\mathcal{L}^p$, and $\mathbb{E}(X_n) \to \mathbb{E}(X)$.*

Proof: needs Monotone Convergence Theorem and Fatou's Lemma.
See page 36, in van Handel's note.

see pg

Theorem 1.5.10 (Monotone convergence). *Let $\{X_n\}$ be a sequence of random variables such that $0 \le X_1 \le X_2 \le \ldots$ a.s. Then $\mathbb{E}(X_n) \nearrow \mathbb{E}(\lim_{n\to\infty} X_n)$.*

↳ pointwise!

Lemma 1.5.11 (Fatou's lemma). *Let $\{X_n\}$ be a sequence of a.s. nonnegative random variables. Then $\mathbb{E}(\liminf X_n) \le \liminf \mathbb{E}(X_n)$. In there exists a Y such that $X_n \le Y$ a.s. for all n and $\mathbb{E}(Y) < \infty$, then $\mathbb{E}(\limsup X_n) \ge \limsup \mathbb{E}(X_n)$.*

But, the usefulness of DCT is that often $\{x_n\}$ is a seq. of bounded r.v.; thus, $x_n \to x$ in prob $\Rightarrow$ $x_n \to x$ in any $\mathcal{L}^p$ and $E[x_n] \to E[x]$.

★ We like $\mathcal{L}^p$ convergence because, it is strong enough and also is a "complete space"; i.e.

if $\{x_n\}$ is Cauchy in $\mathcal{L}^p$ ($\|x_n - x_m\|_{\mathcal{L}^p} \to 0$ as $n,m\to\infty$)

Then, $\exists\, X \in \mathcal{L}^p$ s.t. $X_n \to X$, in $\mathcal{L}^p$.

★ the strong convergence enables us to argue about sample pathes $t \mapsto X_t(\omega)$ for each fixed $\omega \in \Omega$.

Summary of Convergence modes:

stronger ←- - - - - - - - - - - → weaker

$X_n \to X$ a.s.

"If $P(|X_n - X| > \varepsilon) \to 0$ fast (summable)"

$X_n \to X$, in prob $\Rightarrow$ $X_n \to X$ in Law

"if x_n is Cauchy in prob."

$X_n \to X$ in $\mathcal{L}^p$

"If $\{X_n\}$ is dominated in $\mathcal{L}^p$"

Induced measures and laws:

Definition 1.6.1. If $h : \Omega \to S$ is measurable, then the map $\mu : \mathcal{S} \to [0, 1]$ defined by $\mu(A) = \mathbb{P}(h^{-1}(A))$ is a probability measure (why?), called the *induced measure*.

If h is a r.v. then μ_h on $\mathcal{B}$ is called the law of h.

Definition 1.6.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

1. A countable set of events $\{A_n\}_{n \in I} \subset \mathcal{F}$ are *independent* if

$$\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_n}) = \mathbb{P}(A_{k_1})\,\mathbb{P}(A_{k_2}) \cdots \mathbb{P}(A_{k_n}),$$

for all finite ($n < \infty$) subsets $\{k_1, \ldots, k_n\}$ of the index set I.

2. A countable set $\{\mathcal{G}_n\}_{n \in I}$ of σ-algebras $\mathcal{G}_n \subset \mathcal{F}$ are *independent* if any finite set of events $A_1, \ldots, A_n$ from distinct $\mathcal{G}_k$ are independent.

3. A countable set $\{X_n\}_{n \in I}$ of random variables are *independent* if the σ-algebras generated by these random variables $\mathcal{X}_n = \sigma(X_n)$ are independent.

Definition 1.6.5. Given two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, we define the *product space* $\Omega_1 \times \Omega_2 \equiv \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}$. The *product σ-algebra* on $\Omega_1 \times \Omega_2$ is defined as $\mathcal{F}_1 \times \mathcal{F}_2 \equiv \sigma\{\rho_1, \rho_2\}$, where $\rho_i : \Omega_1 \times \Omega_2 \to \Omega_i$ are the projection maps $\rho_1 : (\omega_1, \omega_2) \mapsto \omega_1$ and $\rho_2 : (\omega_1, \omega_2) \mapsto \omega_2$.

Theorem 1.6.6. *Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be two probability spaces, and denote by $\rho_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ and $\rho_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ the projection maps.*

1. *There exists a unique probability measure $\mathbb{P}_1 \times \mathbb{P}_2$ on $\mathcal{F}_1 \times \mathcal{F}_2$, called the product measure, under which the law of ρ_1 is $\mathbb{P}_1$, the law of ρ_2 is $\mathbb{P}_2$, and ρ_1 and ρ_2 are independent Ω_1- and Ω_2-valued random variables, respectively.*

2. *The construction is in some sense unique: suppose that on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are defined an S-valued random variable X (with law μ_X) and a T-valued random variable Y (with law μ_Y). Then the law of the $S \times T$-valued random variable (X, Y) is the product measure $\mu_X \times \mu_Y$.*

3. *If $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $\mathcal{F}_1 \times \mathcal{F}_2$-measurable, then $f(\omega_1, \cdot) : \Omega_2 \to \mathbb{R}$ is $\mathcal{F}_2$-measurable for all ω_1, and $f(\cdot, \omega_2) : \Omega_1 \to \mathbb{R}$ is $\mathcal{F}_1$-measurable for all ω_2.*

4. ***(Tonelli)** If $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $\mathcal{F}_1 \times \mathcal{F}_2$-measurable and $f \geq 0$ a.s., then*

$$\mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}_2}(f) = \mathbb{E}_{\mathbb{P}_2}\left[\int f(\omega_1, \omega_2)\,\mathbb{P}_1(d\omega_1)\right] = \mathbb{E}_{\mathbb{P}_1}\left[\int f(\omega_1, \omega_2)\,\mathbb{P}_2(d\omega_2)\right].$$

In particular, these expressions make sense (the inner integrals are measurable).

5. ***(Fubini)** The previous statement still holds for random variables f that are not necessarily nonnegative, provided that $\mathbb{E}(|f|) < \infty$.*

Tonelli and Fubini theorems are useful for interpreting

$$\mathbb{E}_{\mathbb{P}_1 \times \mathbb{P}_2}[f(X_1, X_2)] = E_{\mathbb{P}_1}[E_{\mathbb{P}_2}[f(X_1, X_2)]]$$

Stochastic Process: $\{X_t,\ t \in [0,T]\}$ is

a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ if each X_t is a r.v.

— Such a process is called measurable ~~ra~~

$X : [0,T] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,T]) \times \mathcal{F}$ - measurable

So, $Y : \Omega \to \mathbb{R}$, $Y(\omega) = \int_0^T X_t(\omega)\, dt$ is a well-defined r.v.

Now, under mild assumptions of Fubini or Tonelli, we get

$$\mathbb{E}[Y] = \mathbb{E}\left[\int_0^T X_t(\omega)\, dt\right] = \int_0^T E[X_t]\, dt \quad !$$

— Next time we do

- Conditional Expectation
- Martingales
- Stochastic Processes

Recall that we think of the Wiener process as the limit as $N \to \infty$, in a suitable sense, of the random walk

$$x_t(N) = \sum_{n=1}^{\lfloor Nt \rfloor} \frac{\xi_n}{\sqrt{N}},$$

where ξ_n are i.i.d. random variables with zero mean and unit variance. That there

Lemma 3.1.1 (Finite dimensional distributions). *For any finite set of times $t_1 < t_2 < \cdots < t_n$, $n < \infty$, the n-dimensional random variable $(x_{t_1}(N), \ldots, x_{t_n}(N))$ converges in law as $N \to \infty$ to an n-dimensional random variable $(x_{t_1}, \ldots, x_{t_n})$ such that $x_{t_1}, x_{t_2} - x_{t_1}, \ldots, x_{t_n} - x_{t_{n-1}}$ are independent Gaussian random variables with zero mean and variance $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$, respectively.*

Definition 3.1.3. A stochastic process W_t is called a *Wiener process* if

1. the finite dimensional distributions of W_t are those of lemma 3.1.1; and
2. the sample paths of W_t are continuous.