

Suppose $X_t \in \mathbb{R}^n$ is a Itô process:

(dynamics) $dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, t \geq 0.$

Suppose we have continuous measurements $H_t: H_t = C(t, X_t) + \gamma(t, X_t) \cdot \tilde{W}_t \leftarrow \text{noise}$

→ we model this by a process $Z_t \in \mathbb{R}^m$:

(observation) $dZ_t = C(t, X_t) dt + \gamma(t, X_t) dV_t, Z_0 = 0$

→ so $Z_t = \int_0^t H_s ds$

V_t, W_t are independent Brownian motion.

In general: $b: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, \sigma: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n \times p}$
 $C: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m, \gamma: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{m \times r}$

$V_t: r\text{-dim.}, W_t: p\text{-dim.}$ Brownian motion.

Filtering problem:

Given $\{Z_s: 0 \leq s \leq t\}$, what is the best estimate $\hat{X}_t$ of the state X_t ?

1) $\hat{X}_t \in G_t = \sigma(\{Z_s: 0 \leq s \leq t\})$

2) Best in mean squared Error (MSE) sense:

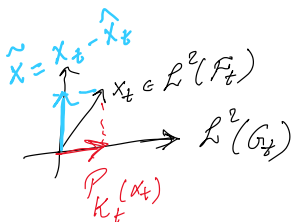
$$\int_{\Omega} |X_t - \hat{X}_t|^2 dP = E[|X_t - \hat{X}_t|^2] = \inf_{Y \in L^2(G_t)} E[|X_t - Y|^2]$$

$$L^2(G_t) := \{Y: \Omega \rightarrow \mathbb{R}^n; Y \in L^2(P), Y \in G_t\}$$

Here, we always assume (Ω, F, P) is the probability space corresponding to (W_t, V_t) .

Thus, by projection Lemma:

$$\hat{X}_t = P_{L^2(G_t)}(X_t) = E[X_t | G_t]$$



Why? because $\forall y \in L^2(G_t) : 0 = \langle X_t - P_{L^2(G_t)}(X_t), y \rangle = \int_{\Omega} y (X_t - P_{L^2(G_t)}(X_t)) dP$.

In particular, $\forall A \in G_t : \int_A [X_t - P_{L^2(G_t)}(X_t)] dP = 0$

which implies $\int_A X_t dP = \int_A P_{L^2(G_t)}(X_t) dP$.

So, by uniqueness of Kolmogorov definition of conditional expectation

$$P_{L^2(G_t)}(X_t) = E[X_t | G_t] .$$

How do we compute this?

1-dim linear filtering problem:

$$\begin{array}{l} \text{(linear dyn.)} \\ \text{(linear observations.)} \end{array} \quad \boxed{\begin{array}{l} dX_t = F(t) X_t dt + C(t) dW_t , \\ dZ_t = G(t) X_t dt + D(t) dV_t , \end{array}} \quad \left\{ \begin{array}{l} \text{in general} \\ F \in \mathbb{R}^{n \times n} \\ C \in \mathbb{R}^{n \times p} \\ G \in \mathbb{R}^{m \times n} \\ D \in \mathbb{R}^{m \times r} \end{array} \right.$$

Assumptions:

→ let's assume $Z_0 = 0$, and $X_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and independent of W_t, V_t .

and F, C, G, D are all bounded on finite intervals and

and D is bounded away from zero on finite intervals ($D_t^T D_t > 0$).

next, we solve this filtering problem in 5 steps:

Step 1: The best MSE estimate is linear in Z_t :

$$\hat{X}_t = P_{L^2(G_t)}(X_t) = E[X_t | G_t] = P_{L(Z,t)}(X_t)$$

where $L(Z,t) \subset L^2(P)$ is the class of functions which are linear combinations of the form:

$$C_0 + C_1 Z_{s_1}(w) + \dots + C_k Z_{s_k}(w) , \quad s_j \leq t, C_j \in \mathbb{R}, \quad k\text{-finite.}$$

Step 2: Replace Z_t with the innovation process N_t :

$$N_t = Z_t - \int_0^t P_{L(Z,s)}(G_s X_s) ds$$

$$= Z_t - \int_0^t G_s \hat{X}_s ds$$

properties of N_t : 1) N_t has orthogonal increments:

$$\text{if } [s_1, t_1] \cap [s_2, t_2] = \emptyset \Rightarrow E[(N_{t_1} - N_{s_1})(N_{t_2} - N_{s_2})] = 0$$

$$2) L(N, t) = L(Z, t) \text{ so } \hat{X}_t = P_{L(N,t)}(X_t).$$

Step 3: Define $dR_t = \frac{1}{D(t)} dN_t$

then R_t is a 1-dim. Brownian motion. And,

$$L(N, t) = L(R, t) \text{ and}$$

$$\hat{X}_t = P_{L(N,t)}(X_t) = P_{L(R,t)}(X_t) = E[X_t] + \int_0^t \frac{\partial}{\partial s} E[X_t R_s] dR_s.$$

Step 4: Solve X_t in closed form: for any $r \in [0, t]$

$$X_t = \exp\left(\int_r^t F(s) ds\right) X_r + \int_r^t \exp\left(\int_s^t F(u) du\right) C(s) dW_s$$

$$\Rightarrow E[X_t] = \exp\left(\int_r^t F(s) ds\right) E[X_r].$$

Step 5: Substitute X_t from step 4 into $E[X_t R_s]$ and use step 3

to obtain the SDE for $\hat{X}_t$:

$$d\hat{X}_t = \frac{\partial}{\partial s} E[X_t R_s] \Big|_{s=t} \cdot dR_t + \left(\int_0^t \frac{\partial^2}{\partial t \partial s} E[X_t R_s] dR_s \right) dt + \dots$$

Step 1: We show that the linear estimator is MSE optimal bc of Gaussianity:

let $\{X, Z_s, s \leq t\}$ be all in $L^2(P)$ s.t.

$(X, Z_{s_1}, \dots, Z_{s_n})$ is jointly normal $\forall s_k \leq t, n \geq 1$

and let $\tilde{X} = X - P_{L(Z,t)}(X)$. then we claim that

$$\tilde{X} \perp G_t$$

→ we use these facts about Gaussians:

(1) $(Y_1, \dots, Y_n)$ is Gaussian $\Leftrightarrow \sum c_i Y_i$ is Gaussian $\forall c_i \in \mathbb{R}$.

(2) L^2 -limit of Gaussians is a Gaussian r.v.

(3) Uncorrelated Gaussian r.v.s are independent.

We can say that

by (1) → $(\tilde{X}, Z_{s_1}, \dots, Z_{s_n})$ is jointly Gaussian.

$$\begin{aligned} \text{linear in } \{Z_s\} \left(\sum_{s_j \leq t} z_{sj} \right) &\rightarrow \mathbb{E}[\tilde{X} Z_{sj}] = \mathbb{E}[(X - \mathbb{E}[X | L(Z,t)]) Z_{sj}] \\ &= \mathbb{E}[X Z_{sj} - \mathbb{E}[X Z_{sj} | L(Z,t)]] \\ &= 0 \quad \Rightarrow \tilde{X}, Z_{sj} \text{ are uncorrelated} \end{aligned}$$

(by tower) →

by (3) → $\tilde{X} \perp (Z_{s_1}, \dots, Z_{s_j})$ independent!

$$\text{Now, } \mathbb{E}[1_A \tilde{X}] = \mathbb{E}[1_A] \cdot \mathbb{E}[\tilde{X}] = 0, \quad \forall A \in \underline{G_t}$$

← any function of $\{Z_s\}$

$$\text{implying that } \mathbb{E}[1_A X] = \mathbb{E}[1_A P_{L(Z,t)}(X)]$$

implying

and also that $P_{L(z,t)}(x) \in G_t$

So, by Kolmogorov uniqueness of cond. exp. $E[x | G_t] = P_{L(z,t)}(x)$.

Finally, to apply this result to our problem, we need to

argue that $M_t = \begin{bmatrix} x_t \\ z_t \end{bmatrix}$ is a Gaussian process:

$$dM_t = H(t) M_t dt + K(t) dB_t \quad \text{with } B_t \text{ 2-dim Brownian}$$

and with $M_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \rightarrow \text{Gaussian}$.

Because x_0 is Gaussian and the dynamic is linear:

$$(\text{Picard's iteration}) \quad M_t^{(n)} = M_0 + \int_0^t H_s M_s^{(n-1)} ds + \int_0^t K(s) dB_s$$

each $M_t^{(n)}$ is Gaussian and $M_t^{(n)} \rightarrow M_t$ in $L^2(\mathbb{P})$.

so by (2), M_t is Gaussian. ~~end~~ end of step 1.

End of
Lecture 12

Step 2: Before talking about the Innovation process, we argue

Lemma 1: $\mathcal{L}(z,t) = \left\{ c_0 + \int_0^t f(s) dz_s : f \in L^2[0,t], c_0 \in \mathbb{R} \right\} \triangleq \mathcal{I}(z,t)$

limits of finite linear combinations of z_t .

Proof: 1) $\mathcal{I}(z,t) \subset \mathcal{L}(z,t)$ because

$$c_0 + \int_0^t f(s) dz_s = L^2\text{-limit}_{\{\Delta t_i\} \rightarrow 0} c_0 + \sum_i f(t_i) \Delta z_i$$

2) Any finite linear combination of $z_i \in \mathcal{I}(z,t)$ because

$$\sum_i c_i z_i = \sum_i c_i' \Delta z_i = \sum_i \int_{t_i}^{t_{i+1}} c_i' dz_t = \int_0^t \sum_i c_i' 1_{[t_i, t_{i+1})}^{(t)} dz_t$$

3) $I(z, t)$ is closed in $L^2(P)$ and thus $L(z, t) \subset I(z, t)$

because: $\exists c_1, c_2 > 0$ s.t. $\forall f \in L^2[0, T]$

$$c_1 \int_0^T f^2(t) dt \leq \mathbb{E} \left[\left(\int_0^T f(t) dz_t \right)^2 \right] \leq c_2 \int_0^T f^2(t) dt \text{ and } L^2[0, T] \text{ is complete.}$$

The Innovation process N_t :

$$\begin{aligned} N_t &= Z_t - \int_0^t \mathcal{P}_{L(z, s)} (G_s X_s) ds \\ &= Z_t - \int_0^t G_s \hat{X}_s ds \quad \text{bc } \underline{G_s \text{ is just a func. of time.}} \end{aligned}$$

so $dN_t = G_t (X_t - \hat{X}_t) dt + D_t dV_t$

Thm:

1) N_t has orthogonal increments

2) $\mathbb{E}[N_t^2] = \int_0^t D_s^2 ds$

3) $L(N, t) = L(Z, t) \quad \forall t \geq 0$

4) N_t is a Gaussian process

proof: $\rightarrow s \leq r \leq t, \quad y \in L(z, s) \subset L(z, r)$

$$\begin{aligned} \mathbb{E}[(N_t - N_s) \cdot y] &= \mathbb{E} \left[\left(\int_s^t G_r (X_r - \hat{X}_r) dr + \int_s^t D_r dV_r \right) y \right] \\ &= \int_s^t G_r \underbrace{\mathbb{E}[(X_r - \hat{X}_r) y]}_{X_r - \hat{X}_r \perp L(z, r)} dr + \underbrace{\mathbb{E} \left[\left(\int_s^t D_r dV_r \right) y \right]}_{V_r \perp y} = 0 \quad \checkmark \end{aligned}$$

$$\rightarrow d(N_t^2) = 2N_t dN_t + \frac{1}{2} \cdot 2 (dN_t)^2 = 2N_t dN_t + D_t^2 dt$$

$$E[N_t^2] = E\left[\int_0^t 2N_s dN_s\right] + \int_0^t D_s^2 ds$$

$$E\left[\lim_{\downarrow} \sum_i 2N_{t_i} \Delta N_i\right] = \lim_{\downarrow} \sum_i E[N_{t_i} \Delta N_i] = 0$$

$N_{t_i} \perp \Delta N_i$
indep. increments

$\rightarrow$ It's clear that $\mathcal{L}(N, t) \subset \mathcal{L}(Z, t) \quad \forall t$. The converse is non-trivial!
 N_t is a linear combination of $\{Z_t\}$.

We wanna show that any Z_s can be written as Lin. Comb. of $\{N_t\}$:

for any $f \in L^2[0, t]$

$$\begin{aligned} \int_0^t f_s dN_s &= \int_0^t f_s dZ_s - \int_0^t f_s G_s \hat{X}_s ds & \begin{cases} \hat{X}_s \in \mathcal{L}(Z, s) \\ \Downarrow \\ \hat{X}_s = C_0(s) + \int_0^s g(r) dz_r \end{cases} \\ &= \int_0^t f_s dZ_s - \int_0^t f_s G_s \int_0^s g(r) dz_r ds - \int_0^t f_s G_s C_0(s) ds & \begin{matrix} \text{triangle} \\ 0 \end{matrix} \\ &= \int_0^t f_r dz_r - \int_0^t \int_s^t f_s G_s g(r) ds dz_r - \int_0^t f_s G_s C_0(s) ds & \begin{matrix} \text{triangle} \\ t \end{matrix} \\ &= \int_0^t \underbrace{\left[f_r - \int_s^t f_s G_s g(r) ds \right]}_{\text{"Volterra Equation" can be = any } h \in L^2[0, t]} dz_r - \int_0^t f_s G_s C_0(s) ds \end{aligned}$$

choose $h = \chi_{[0, t]}$ $\Rightarrow Z_{t_1} = \int_0^t f_s dN_s + \int_0^t f_s G_s C_0(s) ds \quad \checkmark$

$\rightarrow$ Finally, $N_t = Z_t - \int_0^t G(s) \hat{X}_s ds$ and $\hat{X}_s = C_0 + \int_0^s g(s) dz_s$
 $\nwarrow \hat{X}_s \in \mathcal{L}(Z, s)$

$\{Z_t\}$ is a Gaussian process and $\hat{X}_s$ is the L_2 -limits of linear combination of Gaussians $\Rightarrow \hat{X}_s$ is Gaussian $\Rightarrow N_t$ is Gauss.

step 3 : $dN_t = dz_t - G_s \hat{x}_s ds$

Define $dR_t = \frac{1}{D_t} \cdot dN_t$, $t \geq 0$, $R_0 = 0$

Then : If $D_t > \varepsilon > 0, \forall t$ then R_t is a 1-dim Brownian motion!

Proof: R_t is Gaussian process with independent increments
and has continuous path because N_t is.

$$R_t = \int_0^t \frac{1}{D_s} dN_s \quad , \quad dN_t = G_t(x_t - \hat{x}_t) dt + D_t dV_t$$

$$= \int_0^t \frac{G_s}{D_s} (x_s - \hat{x}_s) ds + \int_0^t dV_s$$

$$\Rightarrow E[R_t] = \int_0^t \frac{G_s}{D_s} \underbrace{E[x_s - \hat{x}_s]}_{=0} ds + 0 = 0 \quad \text{by tower property}$$

$E[R_t^2] = t \Rightarrow$ by Ito Formula:

$$d(R_t^2) = 2R_t dR_t + \frac{1}{2} \cdot 2 \cdot (dR_t)^2$$

$$= 2R_t dR_t + dt \quad \Rightarrow R_t^2 = \int_0^t 2R_s dR_s + t$$

$$\Rightarrow E[R_t^2] = t \quad \checkmark$$

$$E[R_t R_s] = E[(R_t - R_s) R_s] + E[R_s^2] = s \quad \text{if } t \geq s.$$

$\Rightarrow R_t$ is a 1-d Brownian motion!

End of
Lecture 13

Note that $dR_t = \underbrace{\frac{G_s}{D_s} (X_t - \hat{X}_t)}_{\neq 0 \text{ but } \mathbb{E}[X_t - \hat{X}_t | \mathcal{F}_t^R] = 0} dt + dV_t$

Thm: $dV_t = a(t, \omega) dt + \sigma(t, \omega) dV_t$
 V_t is a Brownian motion $\Leftrightarrow \mathbb{E}[a(t, \cdot) | \mathcal{F}_t^V] = 0$ and $\sigma \sigma^T(t, \omega) = I$, a.a. (t, ω) .

$\rightarrow$ Next, clearly $\mathcal{L}(V, t) = \mathcal{L}(R, t)$ so

$$\hat{X}_t = P_{\mathcal{L}(R, t)}(X_t) = C_0(t) + \int_0^t g(s) dR_s \quad (\text{by lemma 1})$$

for some $g(s) \in L^2[0, t]$.

$$\rightarrow C_0(t) = \mathbb{E}[\hat{X}_t] = \mathbb{E}[\mathbb{E}[X_t | \mathcal{F}_t^R]] = \mathbb{E}[X_t] \quad (\text{by tower prop.})$$

$\rightarrow$ let's find $g(s)$: by projection lemma

$$X_t - \hat{X}_t \perp \int_0^t f(s) dR_s \quad \forall f \in L^2[0, t].$$

$$\begin{aligned} \text{So, } \mathbb{E}\left[X_t \int_0^t f(s) dR_s\right] &= \mathbb{E}\left[\hat{X}_t \int_0^t f(s) dR_s\right] \\ &= \mathbb{E}\left[\int_0^t g(s) dR_s \int_0^t f(s) dR_s\right] = \int_0^t g(s) f(s) ds \quad (\text{Ito Isometry}) \end{aligned}$$

If we choose $f = \chi_{[0, r]}$ for any $r \leq t$:

$$\mathbb{E}[X_t R_r] = \int_0^r g(s) ds \Rightarrow g(r) = \frac{\partial}{\partial r} \mathbb{E}[X_t R_r].$$

Step 4: Use the integrating factor $Y_t = \exp(-\int_0^t F(s) ds) X_t$

to solve $dX_t = F(s) dt + C(s) dW_s$ and get

$$X_t = \underbrace{\exp\left(\int_0^t F(s) ds\right)}_{\triangleq \varphi(0,t)} \left[X_0 + \int_0^t \exp\left(-\int_0^s F(u) du\right) C(s) dW_s \right]$$

$$= \varphi(0,t) X_0 + \int_0^t \varphi(0,t) \cdot \varphi'(0,s) C(s) dW_s$$

(Verify!) $= \varphi(0,t) X_0 + \int_0^t \varphi(s,t) C(s) dW_s$

(Exercise) In general:

$$X_t = \varphi(r,t) X_r + \int_r^t \varphi(s,t) C_s dW_s, \quad 0 \leq r \leq t.$$

Step 5: So far we have that

$$\hat{X}_t = \mathbb{E}[X_t] + \int_0^t \frac{\partial}{\partial s} \mathbb{E}[X_t R_s] dR_s$$

$$dR_r = \frac{G_r}{D_r} (X_r - \hat{X}_r) dr + dV_r$$

So, $\mathbb{E}[X_t R_s] = \int_0^s \frac{G_r}{D_r} \mathbb{E}[X_t \tilde{X}_r] dr$ with $\tilde{X}_r = X_r - \hat{X}_r$
the estimation error!

$$\mathbb{E}[X_t \tilde{X}_r] = \varphi(r,t) \mathbb{E}[X_r \tilde{X}_r] + \underbrace{\mathbb{E}\left[\int_r^t \varphi(s,t) C_s dW_s \cdot \tilde{X}_r\right]}_{=0 \Leftarrow \mathbb{E}[\tilde{X}] = 0, \tilde{X}_r \perp W_s \text{ for } s \in (r,t]}$$

$$= \varphi(r, t) \underbrace{IE[\tilde{x}_r^2]}_{S_r} \quad \text{as } \hat{x}_r \perp \tilde{x}_r \text{ by projection lemma}$$

$$\triangleq \varphi(r, t) S_r$$

so $IE[X_t R_s] = \int_0^s \frac{G_r}{D_r} \cdot \varphi(r, t) S_r dr$ and thus

$$\frac{\partial}{\partial s} IE[X_t R_s] = \frac{G_s}{D_s} \varphi(s, t) S_s \quad \text{and so}$$

$$\boxed{\hat{X}_t = \underbrace{IE[X_t]}_{C_0(t)} + \int_0^t \underbrace{\frac{G_s}{D_s} \varphi(s, t) S_s}_{f(s, t)} dR_s}$$

now, it is left to find $d\hat{x}_t$ explicitly!

$$\rightarrow IE[X_t] = \varphi(0, t) IE[X_0] = IE[\hat{X}_t] \quad \checkmark$$

$$\rightarrow S_t = IE[\tilde{x}_t^2] = IE[X_t^2] - IE[\hat{X}_t^2] \quad \text{by } \begin{array}{c} x_t \\ \nearrow \\ \hat{x}_t \end{array} \begin{array}{c} \nearrow \\ \tilde{x}_t \end{array}$$

$$= IE[X_t^2] - E\left[(\hat{X}_t - IE[\hat{X}_t])^2\right] - IE[\hat{X}_t]^2$$

now, by $IE[\hat{x}_t] = IE[X_t]$:

$$S_t = IE[X_t^2] - IE[(\hat{X}_t - IE[X_t])^2] - IE[X_t]^2$$

by Ito's isometry $= IE[X_t^2] - \int_0^t f^2(s, t) ds - IE[X_t]^2$

$$\overline{E[X_t^2]} = E[\varphi^2(0,t) X_0^2] + E\left[\left(\int_0^t \varphi(s,t) C_s dw_s\right)^2\right] \quad \text{as } X_0 \perp W_s$$

by Ito
Isometry

$$= \varphi^2(0,t) E[X_0^2] + \int_0^t \varphi^2(s,t) C_s^2 ds$$

$$\text{So } S_t = \varphi^2(0,t) [E[X_t^2] - E[X_0^2]] + \int_0^t [\varphi^2(s,t) C_s^2 - f^2(s,t)] ds$$

Recall $\frac{d}{dt} \left[\int_0^t g(s,t) ds \right] = g(t,t) + \int_0^t \frac{\partial}{\partial t} g(s,t) ds$

$$\Rightarrow \frac{dS_t}{dt} = 2 F_t \varphi(0,t) [E(X_t^2) - E(X_0^2)] + \varphi^2(t,t) C_t^2 - f^2(t,t) \\ + \int_0^t [2 F_t \varphi(s,t) C_s^2 - 2 f(s,t) \cdot \frac{\partial f}{\partial t}(s,t)] ds$$

$$\text{but } f(s,t) = \frac{G_s}{D_s} \varphi(s,t) S_s \rightarrow \begin{cases} f(t,t) = \frac{G_t}{D_t} S_t \quad \text{and} \\ \frac{\partial}{\partial t} f(s,t) = \frac{G_s}{D_s} F_t \varphi(s,t) S_s \\ = F_t \cdot f(s,t) \end{cases}$$

$$\text{and } C_0(t) = E[X_t] = \varphi(0,t) E[X_0] \rightarrow C_0'(t) = F_t \varphi(0,t) E[X_0] \\ = F_t \cdot C_0(t)$$

$$S_0: \frac{dS_t}{dt} = 2F_t \underbrace{\varphi(\cdot, t) [E(X_0^2) - E(X_0)^2]} + C_t^2 - \frac{G_t^2}{D_t^2} S_t^2 \\ + 2F_t \int_0^t \underbrace{[\varphi(s, t) C_s^2 - f(s, t)]} ds$$

$$\text{by } S_t = 2F_t S_t + C_t^2 - \frac{G_t^2}{D_t^2} S_t^2$$

Riccati Equation:
$$\dot{S}_t = 2F_t S_t + C_t^2 - \frac{G_t^2}{D_t^2} S_t^2$$

Back to deriving $d\hat{X}_t$: $\hat{X}_t \in \mathcal{L}(R, t) = \mathcal{I}(R, t)$

→ for each fixed t

$$\mathcal{I}(R, t) = \left\{ C_0 + \int_0^t f(s) dR_s : f \in L^2(0, t), C_0 \in \mathbb{R} \right\}$$

→ So

$$\hat{X}_t(\omega) = C_0(t) + \int_0^t f(s, t) dR_s(\omega) \rightsquigarrow \text{Now } d\hat{X}_t = ?$$

Let's compute:

$$\begin{aligned} \int_0^u \left(\int_0^t \frac{\partial}{\partial t} f(s, t) dR_s \right) dt &= \int_0^u \left(\int_s^u \frac{\partial}{\partial t} f(s, t) dt \right) dR_s \\ &= \int_0^u [f(s, u) - f(s, s)] dR_s = \int_0^u f(s, u) dR_s - \int_0^u f(s, s) dR_s \\ &= \hat{X}_u - C_0(u) - \int_0^u f(s, s) dR_s \end{aligned}$$

Thus, $\frac{d}{du} \big|_{u=t} :$

$$\begin{aligned}
 d\hat{x}_t &= c'_0(t)dt + f(t,t) dR_t + \left(\int_0^t \frac{\partial}{\partial t} f(s,t) dR_s \right) dt \\
 &= c'_0(t)dt + \frac{G_t}{D_t} S_t dR_t + \left(\int_0^t F_t \cdot f(s,t) dR_s \right) dt \\
 &= \underbrace{c'_0(t)}_{F_t c_0(t)} dt + \frac{G_t}{D_t} S_t dR_t + \underbrace{\left(\int_0^t f(s,t) dR_s \right)}_{\hat{x}_t - c_0(t)} \cdot F_t dt \\
 &= F_t c_0(t)dt + F_t [\hat{x}_t - c_0(t)] dt + \frac{G_t}{D_t} S_t dR_t \\
 &= F_t \hat{x}_t dt + \frac{G_t}{D_t} S_t dR_t
 \end{aligned}$$

Recall that $dR_t = \frac{1}{D_t} dN_t = \frac{1}{D_t} [dz_t - G_t \hat{x}_t dt]$

So

$$d\hat{x}_t = F_t \hat{x}_t dt + \frac{G_t}{D_t^2} S_t dz_t - \frac{G_t^2}{D_t^2} S_t \hat{x}_t dt$$

Thus, the 1-dim MSE optimal estimator is the Kalman-Bucy filter :

$$d\hat{x}_t = \left[F_t - \frac{G_t^2}{D_t^2} S_t \right] \hat{x}_t dt + \frac{G_t}{D_t^2} S_t dz_t, \quad \hat{x}_0 = E[x_0]$$

where $S_t \triangleq E[\tilde{x}_t^2] = E[(x_t - \hat{x}_t)^2]$ satisfies the Riccati equation

$$\frac{dS_t}{dt} = 2F_t S_t - \frac{G_t^2}{D_t^2} S_t^2 + C_t^2, \quad S_0 = E[(x_0 - E(x_0))^2]$$

Examples: noisy observations of a constant process

(constant dyn) $dx_t = 0$, and x_0 is Gaussian $\begin{cases} \mathbb{E}[x_0] = \mu_0 \\ \mathbb{E}[x_0^2] = a^2 \end{cases}$

(linear observ.) $dz_t = x_t dt + m dv_t$, $z_0 = 0$

$\hookrightarrow$ now, this corresponds to $H_t = \frac{dz_t}{dt} = x_t + m \tilde{W}_t$

but we said that we don't have a well-defined mathematical model of white noise which we wanna think as derivative of a Wiener process " $\frac{dv_t}{dt}$ "! that is why we worked with z_t instead

s.t. $z_t = \int_0^t H_s ds \rightsquigarrow$ essentially a sum over such "white noise"

which is well-defined!

$\rightarrow$ What is the Kalman-Bucy filter?

solve the Riccati equation for $S_t = \mathbb{E}[|x_t - \hat{x}_t|^2]$:

$$\frac{dS_t}{dt} = -\frac{1}{m^2} S_t^2, \quad S(0) = a^2$$

$$\Rightarrow S_t = \frac{a^2 m^2}{m^2 + a^2 t}, \quad t \geq 0$$

So the estimate $\hat{X}_t$:

$$d\hat{X}_t = - \underbrace{\frac{a^2}{m^2 + a^2 t}}_{F_t} \hat{X}_t dt + \frac{a^2}{m^2 + a^2 t} dz_t, \quad \hat{X}_0 = E[X_0]$$

Ex: Use the integrating factor $\exp(-\int_0^t F_s ds)$ to get:

$$\hat{X}_t = \frac{m^2}{m^2 + a^2 t} \hat{X}_0 + \frac{a^2}{m^2 + a^2 t} z_t, \quad t \geq 0, \quad \hat{X}_0 = E[X_0]$$