Lecture 11

Tuesday, February 10, 2026 5:45 PM

Review:

$$dX_t = b(X_t, t)\cdot dt + \sigma(X_t, t)\cdot dW_t$$

where W_t is the standard wiener process

Existence and uniqueness:

(Growth Cond.) $\exists\, C<\infty :\ |b(x,t)| + |\sigma(x,t)| \le C(1+|x|) \quad \forall x,\ \forall t\in[0,T]$

(Lipschitz Cond.) $\exists\, L<\infty :\ |b(x,t)-b(x',t)| + |\sigma(x,t)-\sigma(x',t)| \le L|x-x'| \quad \forall x,x',\ t\in[0,T]$

→ we can replace the growth condition in this theorem by (see last lecture)

$$\exists\, M>0 :\ \sup_{t\in[0,T]} \left(|b(0,t)| + |\sigma(0,t)|\right) < M .$$

Markov property:

Recall that $\{W_t : t\ge 0\}$ is a markov process. Now, define

$\{\hat{W}_r = W_{r+s} - W_s : r\ge 0\} \rightsquigarrow$ wiener process $\perp F_s^{\infty}$.

Consider the following two Itô processes:

$$X_t = X_0 + \int_0^t b(X_r, r)\,dr + \int_0^t \sigma(X_r, r)\,dW_r$$

$$= X_s + \int_s^t b(X_r, r)\,dr + \int_s^t \sigma(X_r, r)\,dW_r$$

and $$Y_{t-s} = Y_0 + \int_0^{t-s} b(Y_r, r+s)\,dr + \int_0^{t-s} \sigma(Y_r, r-s)\,d\hat{W}_r .$$

Note that $X_s \in \mathcal{F}_s^W$ and $X_t \in \sigma(X_s, W_r : r \in [s,t])$, and

$Y_{t-s} \in \sigma(Y_0, \hat{W}_r : r \in [0, t-s])$.

So if $Y_0 = X_s$, then $Y_t = X_{t+s}$ by uniqueness of sol,

and also $X_t \in \sigma(X_s, (W_{r+s} - W_s) : r \in [0, t-s])$. So, by

Doob-Dynkin's Lemma: $X_t = F(X_s, W_{\cdot+s} - W_s)$. Now, just

like the case of markov property for wiener process, by Fubini's thm

$$\mathbb{E}\left[g(X_t) \mid \mathcal{F}_s\right] = \mathbb{E}\left(g \circ F(x, W_{\cdot+s} - W_s)\right)\Big|_{x = X_s} \quad \forall g \text{ bounded meas.}$$

So RHS is only a func. of $X_s \Rightarrow$ RHS $\in \sigma(X_s)$.

Also, $\mathbb{E}[g(X_t) \mid X_s] = \mathbb{E}\left[\mathbb{E}[g(X_t) \mid \mathcal{F}_s] \mid X_s\right]$ by Tower property.

$= \mathbb{E}[\text{RHS} \mid X_s] = \text{RHS}.$

$\Rightarrow \mathbb{E}[g(X_t) \mid \mathcal{F}_s] = \text{RHS} = \mathbb{E}[g(X_t) \mid X_s]$ ∎

→ In the rest of this part, we provide two new ways to study the solutions of SDEs, namely <u>Kolmogorov Backward Eq.</u> and <u>Kolmogorov Forward Eq.</u> (the dual of KBE).

Also, for simplicity we consider the <u>time-homogeneous SDEs</u>.

<u>Kolmogorov Backward Equation:</u>

markov property $\Rightarrow$ $\mathbb{E}[g(X_t) | \mathcal{F}_s] = f_{t,s}(X_s)$ for some non-random $f_{t,s}(\cdot)$

Suppose $b(x,t) = b(x)$ and $\sigma(x,t) = \sigma(x)$ for simplicity.

then by checking the proof of markov property:

$\mathbb{E}[g(X_t) / \mathcal{F}_s] = f_{t-s}(X_s)$ $\rightsquigarrow$ "time-homogeneous process"

So Instead of studying $\{X_t\}_t$, we can investigate the family $\{f_t\}_t$

Consider the map $P_t : g \longmapsto f_t$ and show that it is a Semi-group

1) P_t is linear : bc. cond. exp. is linear

2) $P_t P_r = P_{t+r}$: $\mathbb{E}[g(X_{t+r+s}) | \mathcal{F}_s] = f_{t+r}(X_s)$ $\leftarrow$ $P_{t+r} g = f_{t+r}$

$\mathbb{E}[g(X_r) | \mathcal{F}_s] = f_{r-s}(X_r)$ $\leftarrow$ $P_r g = f_r$

$f_{t+r}(X_s) = \mathbb{E}[g(X_{t+r+s}) | \mathcal{F}_s] \overset{\text{tower}}{=} \mathbb{E}\left[\mathbb{E}[g(X_{t+r+s}) | \mathcal{F}_s] | \mathcal{F}_r \right]$ $\leftarrow$ $P_t(P_r g)$

So, we have something like: $\frac{d}{dt} P_t = \mathcal{L} P_t$, $P_0 = I$

similar to the semi-group for ODEs. Here is an easier way to state it:

Proposition 5.2.4 (Kolmogorov backward equation). *For $g \in C^2$, define*

$$\mathscr{L} g(x) = \sum_{i=1}^{n} b^i(x) \frac{\partial g}{\partial x^i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{m} \sigma^{ik}(x) \sigma^{jk}(x) \frac{\partial^2 g}{\partial x^i \partial x^j}(x).$$

Suppose that there is a bounded function $u(t,x)$ which is C^1 in t and C^2 in x, and a bounded function $f(x)$ in C^2, such that the following PDE is satisfied:

$$\frac{\partial}{\partial t} u(t,x) = \mathscr{L} u(t,x), \qquad u(0,x) = f(x).$$

Then $\mathbb{E}(f(X_t) | \mathcal{F}_s) = u(t-s, X_s)$ a.s. for all $0 \le s \le t \le T$, i.e., $u(t,x) = P_t f(x)$.

Remark 5.2.5. The operator $\mathscr{L}$ should look extremely familiar—this is precisely the expression that shows up in Itô's rule! Not surprisingly, this is the key to the proof. $\mathscr{L}$ will show up frequently in the rest of the course.

Remark 5.2.6. You might wonder why the above PDE is called the *backward* equation. In fact, we can just as easily write the equation backwards in time: setting $v(t,x) = u(T-t,x)$ and using the chain rule, we obtain

$$\frac{\partial}{\partial t} v(t,x) + \mathscr{L} v(t,x) = 0, \qquad v(T,x) = f(x),$$

which has a terminal condition (at $t = T$) rather than an initial condition (at $t = 0$). For time-nonhomogeneous Markov processes, the latter (backward) form is the appropriate one, so it is in some sense more fundamental. As we have assumed time-homogeneity, however, the two forms are completely equivalent in our case.

Proof: Set $v(r,x) = u(t-r,x)$, and let $Y_r = v(r,X_r)$.

$$dY_r = \frac{\partial v}{\partial r}\cdot dr + \frac{\partial v}{\partial x}\cdot dX_r + \frac{1}{2} dX_t^T \cdot \frac{\partial^2 v}{\partial x^2}\cdot dX_r$$

$$= \left[\frac{\partial v}{\partial r} + \frac{\partial v}{\partial x}\cdot b + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^T \frac{\partial^2 v}{\partial x^2}\right)\right] dr + \frac{\partial v}{\partial x}\cdot\sigma\, dW_r$$

$$= \left[\frac{\partial v}{\partial r} + \mathscr{L} v\right] dr + \frac{\partial v}{\partial x}\cdot\sigma\, dW_r$$

$$r\in[0,t] \Rightarrow v(r,X_r) = v(0,X_0) + \int_0^r \underbrace{\left[\frac{\partial v}{\partial r} + \mathscr{L} v\right]}_{=0 \text{ by def. of } u} dr + \underbrace{\int_0^r \frac{\partial v}{\partial x}\cdot\sigma\, dW_r}_{\text{(local) martingale}}$$

because: $v(r,X_r) = u(t-r,X_r) \Rightarrow \begin{cases} \frac{\partial v}{\partial r}(r,X_r) = -\frac{\partial u}{\partial r}(t-r,X_r) = -\mathscr{L}u(t-r,X_r) = -\mathscr{L}v(r,X_r) \\ v(t,X_r) = u(0,X_r) = f(X_r) \end{cases}$

$\Rightarrow v(t,X_t)$ is a (local) martingale on each finite $[0,T]$:

$\mathbb{E}[v(r,X_r) \mid \mathcal{F}_s] = v(s,X_s)$ on $r\in[0,T]$, $r\geq s$.

and note that $v(t,X_t) = f(X_t)$

So, $s\leq t\leq T$: $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[v(t,X_t) \mid \mathcal{F}_s] = v(s,X_s) = u(t-s,X_s)$ ■

Remark: The requirement of T being finite is critical in this argument, but we can generalize this argument by the idea of local martingale that use a seq. of stopping times $\tau_n \to \infty$.

see: Van Handel (page 124), and (page 101-2).

Kolmogorov Forward Equation: (Fokker-Planck Eq.)

X_t is a $\mathbb{R}^n$-r.v. $\Rightarrow$ (in best case scenario) it has a prob. density P_t:

$$E[f(X_t)] = \int_{\mathbb{R}^n} f(y)\, P_t(y)\, dy \quad , \quad \forall f \text{ bounded + continuous}$$

more generally, we can find a transition density $P_t(x, y)$

$$E[f(X_t) \mid \mathcal{F}_s] = \int_{\mathbb{R}^n} f(y)\, P_{t-s}(X_s, y)\, dy \quad , \quad \forall f \text{ bdd + cont.}$$

$\to$ why is this dual to Kolmogorov Backward Eq.?

by Tower property

density of X_0

$$\underbrace{\int_{\mathbb{R}^n} f(y)\, P_t(y)\, dy}_{\langle f, P_t \rangle} = \mathbb{E}[f(X_t)] = \mathbb{E}\Big[\overbrace{\mathbb{E}[f(X_t) \mid \mathcal{F}_0]}^{= g(X_0) = (P_t f)(X_0)}\Big] = \underbrace{\int_{\mathbb{R}^n} (P_t f)(y) \cdot P_0(y)\, dy}_{\langle P_t f, P_0 \rangle}$$

duality

$\langle \underset{\text{functions}}{f}, \underset{\text{density}}{P_t} \rangle$ $\qquad$ $\langle \underset{\text{function}}{P_t f}, \underset{\text{density}}{P_0} \rangle$

where $\langle \cdot, \cdot \rangle$ is the pairing of functions to densities:

$$\langle f, \mu \rangle = \int f \, d\mu .$$

$\Rightarrow$ How can we find P_t ?!

Proposition 5.2.8 (Kolmogorov forward equation). *Assume that, in addition to the conditions of theorem 5.1.3, $b(x)$ is C^1 and $\sigma(x)$ is C^2. For $\rho \in C^2$, define*

$$\mathscr{L}^* \rho(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x^i}(b^i(x)\rho(x)) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} \frac{\partial^2}{\partial x^i \partial x^j}(\sigma^{ik}(x)\sigma^{jk}(x)\rho(x)).$$

Suppose that the density $p_t(x)$ exists and is C^1 in t, C^2 in x. Then

$$\frac{\partial}{\partial t} p_t(x) = \mathscr{L}^* p_t(x), \qquad t \in [0, T],$$

i.e., the density $p_t(x)$ of X_t must satisfy the Kolmogorov forward equation.

Proof: Fix $f \in C_0^2$ (C^2 with compact support)

$$f(X_t) = f(X_0) + \int_0^t \mathcal{L} f(X_s)\, ds + \int_0^t \underbrace{\frac{\partial f}{\partial x}}_{\text{bounded}} \sigma \, dW_s$$

bounded $\Rightarrow$ the integral is a martingale

Fubini's $\Rightarrow$ $\mathbb{E}[f(X_t)] = \mathbb{E}[f(X_0)] + \int_0^t \mathbb{E}[\mathcal{L} f(X_s)]\, ds$

by def. of densities P_t :

$$\int_{\mathbb{R}^n} f(y) P_t(y)\, dy = \int_{\mathbb{R}^n} f(y) P_0(y)\, dy + \int_0^t \underbrace{\int_{\mathbb{R}^n} \mathcal{L} f(y) P_s(y)\, dy}\, ds$$

by integration by parts in multidim.:

$$\int_{\mathbb{R}^n} \mathcal{L}f(y)\, P_s(y)\, dy = \int_{\mathbb{R}^n} \Big[\underbrace{(b \cdot \nabla_x f)}_{\sum_i b_i \frac{\partial f}{\partial x_i}}(y)\, P_s(y) + \frac{1}{2} \underbrace{Tr(\nabla_x^2 f \sigma \sigma^T)}_{\sum_{i,j,k} \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \sigma_{jk} \sigma_{ik}}(y) \cdot P_s(y) \Big] dy$$

$$= - \int_{\mathbb{R}^n} f(y) \sum_i \frac{\partial}{\partial x_i} (b_i P_s)(y)\, dy$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n} f(y) \sum_{i,j,k} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sigma_{jk} \sigma_{ik} P_s \right)(y)\, dy$$

$$= \int_{\mathbb{R}^n} f(y) \cdot (\mathcal{L}^* P_s)(y)\, dy \qquad \text{by def. } \mathcal{L}^*.$$

By Fubini's, we have $\forall f \in C_0^2$:

$$\int_{\mathbb{R}^n} f(y)\, P_t(y)\, dy = \int_{\mathbb{R}^n} f(y)\, P_0(y)\, dy + \int_{\mathbb{R}^n} f(y) \cdot \int_0^t \mathcal{L}^* P_s(y)\, ds\, dy$$

$$\Rightarrow \langle f(\cdot), \alpha(\cdot) \rangle = 0 \quad \forall f \in C_0^2$$

$$\text{where } \alpha(y) \triangleq P_t(y) - P_0(y) - \int_0^t \mathcal{L}^* P_s(y)\, ds$$

$\Rightarrow$ $\alpha(y) \equiv 0$ for almost all y!

↳ this follows by a standard argument by choosing $f(y) = \alpha(y) \cdot K(y)$, where $K(\cdot) \in C_0^2$ is a bump func. and use continuity of $\alpha(\cdot)$ (because $\mathcal{L}^* P_t(y)$ is cont. in (t, y) by Dom. Conv. thm. and hence locally bounded).

Simulating SDEs: the Euler-Maruyama method

on $[0,T] \rightsquigarrow t_k = k \cdot \frac{T}{p}$ for $k = 0, 1, \dots, p$.

$$X_{t_n} = X_{t_{n-1}} + \int_{t_{n-1}}^{t_n} b(X_s)\, ds + \int_{t_{n-1}}^{t_n} \sigma(X_s)\, dW_s$$

and X_t has continuous path $\rightsquigarrow X_s \approx X_{t_{n-1}}$ for $s \in [t_{n-1}, t_n]$. if $p \gg 1$.

$$\hat{X}_{t_n} = \hat{X}_{t_{n-1}} + b(\hat{X}_{t_{n-1}}) \cdot (t_n - t_{n-1}) + \sigma(\hat{X}_{t_{n-1}}) \cdot (W_{t_n} - W_{t_{n-1}})$$

Fact: We can show that as $p \to \infty$, $\hat{X}_{t_n} \to X_{t_n}$ in the following sense:

$$\max_{0 \le n \le p} \| X_{t_n} - \hat{X}_{t_n} \|_{\mathcal{L}^2(\mathbb{P})} \le C \cdot p^{-1/2}$$

Proof: see page 131 in van Handel.

Exercises: 5.1 - 5.5 - 5.6 - 5.7 - 5.8

5.10 - 5.11 - 5.12 - 5.16

7.1 - 7.2