Lecture 10

Saturday, February 7, 2026 6:00 PM

Stochastic Differential equations:

$$dX_t = b(X_t, t) \cdot dt + \sigma(X_t, t) \cdot dW_t$$

where W_t is the standard wiener process

Example 1: Consider

$$dX_t = \alpha X_t \, dt + \beta X_t \, dW_t, \quad X_0 \text{ is given.}$$

and derive the SDE for $Y_t = g(X_t)$.

Take $g(x, t) = \ln x$

By Itô Formula:

$$dY_t = \frac{\partial g}{\partial t} \cdot dt + \frac{\partial g}{\partial x} \cdot dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \cdot (dX_t)^2$$

$$= 0 + \frac{1}{X_t} \cdot (\alpha X_t \, dt + \beta X_t \, dW_t) + \frac{1}{2} \cdot \frac{-1}{X_t^2} \cdot [\beta^2 X_t^2 \, dt]$$

$$= [\alpha - \tfrac{1}{2}\beta^2] \, dt + \beta \, dW_t$$

Implying that $Y_t = Y_0 + (\alpha - \frac{1}{2}\beta^2) t + \beta W_t$ which in turn

implies $X_t = X_0 \cdot \exp[(\alpha - \frac{1}{2}\beta^2) t + \beta W_t]$.

Exercise: - Simulate this and argue about the role of α and β when you think of X_t as the value of an investment X_0 after t-time on an asset with uncertain return.

- Rigorously argue about different limits of X_t by using ... three cases $\alpha \gtreqless \frac{1}{2}\beta^2$

For the ...

Lemma: (The Law of iterated logarithm)

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t \cdot \log\log t}} = 1 \quad a.s.$$

Example 2: Consider $X_t = W_t$ and $g(t,x) = e^{ix} = \cos(x) + i\sin(x)$

if $Y_t = g(X_t) = \cos(W_t) + i\sin(W_t) \Rightarrow |Y_t| \equiv 1$.

$$dY_t = -\sin(W_t)\,dW_t - \tfrac{1}{2}\cos(W_t)\,dt + i\left[\cos(W_t)\,dW_t - \tfrac{1}{2}\sin(W_t)\,dt\right]$$

$$dY_t = -\tfrac{1}{2}\left[\cos(W_t) + i\sin(W_t)\right]dt$$
$$\left[-\sin(W_t) + i\cos(W_t)\right]dW_t$$

$$dY_t = -\tfrac{1}{2}Y_t\,dt + i\,\overline{Y_t}\,dW_t$$ "Brownian motion on unit circle"

Existence and Uniqueness of solutions:

Recall the two famous non-examples in ODEs:

1) non-existence: $\frac{dX_t}{dt} = X_t^2$, $X_0 = 1 \Rightarrow X_t = \frac{1}{1-t}$, $0 \le t < 1$.

and as $t \to 1^-$, $X_t \to \infty$ (or doesn't exist for $t \ge 1$).

Issue: $x \mapsto x^2$ grows too fast (faster than $x \mapsto |x|$)

2) non-uniqueness: $\frac{dX_t}{dt} = 3X_t^{2/3}$, $X_0 = 0 \Rightarrow$

for any $a > 0$: $X_t = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$ is a solution! (it is not unique).

Issue: $x \mapsto 3x^{2/3}$ is not Lipschitz continuous

"Strong solutions" of SDEs:

Theorem 5.2.1. (Existence and uniqueness theorem for stochastic differential equations).

Let $T > 0$ and $b(\cdot,\cdot): [0,T] \times \mathbf{R}^n \to \mathbf{R}^n, \sigma(\cdot,\cdot): [0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|)\,; \qquad x\in\mathbf{R}^n,\ t\in[0,T] \tag{5.2.1}$$

for some constant C, (where $|\sigma|^2 = \sum|\sigma_{ij}|^2$) and such that

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le D|x-y|\,; \quad x,y\in\mathbf{R}^n,\ t\in[0,T] \tag{5.2.2}$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot)$, $s\ge 0$ and such that

$$E[|Z|^2] < \infty\,.$$

Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t\,, \qquad 0\le t\le T,\ X_0 = Z \tag{5.2.3}$$

has a unique t-continuous solution $X_t(\omega)$ with the property that

$$X_t(\omega) \text{ is adapted to the filtration } \mathcal{F}_t^Z \text{ generated by } Z \text{ and } B_s(\cdot);\ s\le t \tag{5.2.4}$$

and

$$E\Big[\int_0^T |X_t|^2 dt\Big] < \infty\,. \tag{5.2.5}$$

Note: Here uniqueness means if $X_t(\omega)$ and $\hat{X}_t(\omega)$ are two such solutions then $X_t(\omega) = \hat{X}_t(\omega)$ $\forall t\le T$, a.s.

Proof: (Uniqueness)

suppose
$$\begin{cases} dX_t = b(X_t,t)\,dt + \sigma(X_t,t)\,dW_t\,, & X_0 = Z \\ d\hat{X}_t = b(\hat{X}_t,t)\,dt + \sigma(\hat{X}_t,t)\,dW_t\,, & \hat{X}_0 = \hat{Z} \end{cases}$$

Now, lets bound

$$\mathbb{E}\left[|X_t-\hat{X}_t|^2\right] \le 3\,\mathbb{E}\left[|Z-\hat{Z}|^2\right] + 3\,\mathbb{E}\left[\left|\int_0^t b(X_s,s)-b(\hat{X}_s,s)\,ds\right|^2\right]$$
$$+3\,\mathbb{E}\left[\left|\int_0^t [\sigma(X_s,s)-\sigma(\hat{X}_s,s)]\,dW_s\right|^2\right]$$

by $\left\{\begin{array}{l}\text{Itô Isometry,}\\ \left[\int_0^t a_s ds\right]^2 \le t\int_0^t a_s^2 ds\end{array}\right.$ ← $(b_1+\dots+b_n)^2 \le n(b_1^2+\dots+b_n^2)$

$$\le 3\,\mathbb{E}\left[|Z-\hat{Z}|^2\right] + 3t\,\mathbb{E}\left[\int_0^t |b(X_s,s)-b(\hat{X}_s,s)|^2 ds\right]$$
$$+3\,\mathbb{E}\left[\int_0^t |\sigma(X_s,s)-\sigma(\hat{X}_s,s)|^2 ds\right]$$

by Lipschitz assumption
$$\le 3\,\mathbb{E}\left[|Z-\hat{Z}|^2\right] + 3(1+t)D^2\,\mathbb{E}\left[\int_0^t |X_s-\hat{X}_s|^2 ds\right]$$

Tonelli/Fubini
$$= 3\,\mathbb{E}\left[|Z-\hat{Z}|^2\right] + 3(1+t)D^2\int_0^t \mathbb{E}\left[|X_s-\hat{X}_s|^2\right] ds$$

Define $v_t := 3\mathbb{E}\left[|X_t-\hat{X}_t|^2\right]$, $v_0 = 3\mathbb{E}\left[|Z-\hat{Z}|^2\right]$, then
$$v_t \le v_0 + (1+T)D^2\int_0^t v_s\,ds, \quad 0\le t\le T$$

So by Grownwall lemma : $\Rightarrow$ $v_t \le v_0 \exp\left((1+T)D^2 t\right)$, $0\le t\le T$

Now, if $Z=\hat{Z}$ then $v_0=0 \Rightarrow v_t = 0$, $\forall t\ge 0 \Rightarrow$

$$\mathbb{P}\left[\,|X_t(\omega)-\hat{X}_t(\omega)| = 0\right] = 1, \quad \forall t\in[0,T].$$

$\mathbb{Q}$ is rational numbers $\Rightarrow$ $\mathbb{P}\left[\bigcap_{t\in\mathbb{Q}\cap[0,T]}\{|X_t(\omega)-\hat{X}_t(\omega)|=0\}\right] = 1$

$\left.\right\}$ general argument about Const. path processes.

$t\mapsto |X_t(\omega)-\hat{X}_t(\omega)|$ is continuous $\Rightarrow$ $\mathbb{P}\left[\bigcap_{t\in[0,T]}\{|X_t(\omega)-\hat{X}_t(\omega)|=0\}\right] = 1.$

$\Rightarrow X_t = \hat{X}_t \quad \forall t\le T$, <u>a.s.</u>

- For existence, read page 70 in Øksendal.

Discussion on the growth condition for SDE :

... that is uniform in t :

supp. $F(x,t)$ has Global Lipschitz property ...

$\exists\, k>0$: $|F(x,t) - F(x',t)| \le k|x-x'|$ $\quad \forall x,x', \forall t\in[0,T]$

then $\big|\,|F(x,t)| - |F(0,t)|\,\big| \le k|x|$ or

$$|F(x,t)| \le k|x| + \sup_{t\in[0,T]} |F(0,t)|$$

So, if $|F(0,t)|$ is bounded on $t\in[0,T]$ by $M>0$ then

$$|F(x,t)| \le \max(k,M)\,(1+|x|) \qquad \text{"the growth condition"}$$

Therefore, we can replace the growth condition in this theorem by

$$\exists\, M>0: \sup_{t\in[0,T]}\left(|b(0,t)| + |\sigma(0,t)|\right) < M .$$

"weak solution" of SDE:

- X_t is a "strong solution" of $dX_t = b(X_t,t)\,dt + \sigma(X_t,t)\,dW_t$ where X_t is $\mathcal{F}_t^{Z,W}$-adapted for the given wiener process W_t on a given $(\Omega, \mathcal{F}, \mathbb{P})$.

- Now, only given functions $b(x,t)$ and $\sigma(x,t)$, we may find $(\tilde{X}_t, \tilde{W}_t, \{\mathcal{H}_t\})$ s.t. $d\tilde{X}_t = b(\tilde{X}_t,t)\,dt + \sigma(\tilde{X}_t,t)\,d\tilde{W}_t$ and such that $\tilde{X}_t$ is $\mathcal{H}_t$-adapted and $\tilde{W}_t$ is a $\mathcal{H}_t$-wiener process on some $(\Omega, \mathcal{H}, \mathbb{P})$; i.e. $\tilde{W}_t$ is a wiener process and $\tilde{W}_t$ is a $\mathcal{H}_t$-martingale.

  Then, we call $(\tilde{X}_t, \tilde{W}_t, \{\mathcal{H}_t\})$ a "weak solution".

Lemma: Any weak or strong solution under the same conditions of theorem is weakly unique; meaning that $\tilde{X}_t, \hat{X}_t$ have the same Law for all t, (or have identical finite-dimensional distributions).

Fact: 1- Any strong solution is a weak solution!

2- There might exists weak solution where there is no strong solution. ex: $dX_t = \text{sgn}(X_t)\, dW_t$, $X_0 = 0$,

with $\text{sgn}(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}$. (see page 73 in Øksendal)

→ Note that sgn is not Lipschitz Continuous!

Example: $dX_t = A X_t\, dt + B\, dW_t$, where X_0 given, W_t is n-dim..

"Linear time-invariant LTI system"

Use Itô formula: $g(x,t) = \exp(-At)\, x \; : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$

$$Y_t = g(X_t, t) = \exp(-At)\, X_t \quad \Rightarrow Y_0 = X_0$$

and
$$dY_t = \left[-A \exp(-At)\, X_t + \frac{1}{2} \times 0\right] dt + \exp(-At)\, dX_t$$

$$= \exp(-At)\left[-A X_t + dX_t\right] = \exp(-At)\, B\, dW_t$$

$$d Y_t = \exp(\dots)$$

$$\Rightarrow \quad Y_t = Y_0 + \int_0^t \exp(-As)\, B\, dW_s$$

$$\Rightarrow \quad X_t = \exp(At)\, X_0 + \int_0^t \exp(A(t-s))\, B\, dW_s \quad !$$

<u>Exercise:</u> solve $dX_t = (A X_t + H_t)\, dt + B\, dW_t$.

H_t ↳ deterministic

<u>n-dim Ito Formula (compact form):</u> $Y_t = g(X_t, t)$

with $dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$. Then

$$dY_t = \frac{\partial g}{\partial t}\, dt + \nabla_x g^T dX_t + \frac{1}{2} dX_t^T \nabla_x^2 g\, dX_t$$

$$= \left[\frac{\partial g}{\partial t} + \nabla_x g^T b + \frac{1}{2} Tr(\nabla_x^2 g\, dX_t\, dX_t^T)\right] dt + \nabla_x g^T \sigma\, dW_t$$

$$= \left[\frac{\partial g}{\partial t} + \nabla_x g^T b + \frac{1}{2} Tr(\nabla_x^2 g\, \sigma \underbrace{dW_t\, dW_t^T}_{= I\, dt} \sigma^T)\right] dt + \nabla_x g^T \sigma\, dW_t$$

$$= \left[\frac{\partial g}{\partial t} + \nabla_x g^T b + \frac{1}{2} Tr(\nabla_x^2 g\, \sigma \sigma^T)\right] dt + \nabla_x g^T \sigma\, dW_t.$$
