Lecture 1 (MAE 271B, Boelter 5272)

Sunday, January 4, 2026 7:13 PM

– What is a <u>Brownian Motion</u>?

Consider a particle of interest in a fluid medium: (Brown Pollen Experiment)

– What is <u>the path travelled</u> by this particle in unit time?

The sample path is a result of <u>many small</u>, <u>independent</u> collisions of the particle with the fluid molecules [Einstein's argument]; and these collisions must have <u>zero-mean</u> because the overall velocity is zero.

– How can we model this (stochastic) dynamics?

- We know that the <u>mean-square displacement of the particle per unit time</u> must remain constant, say $=\gamma$, and only depends on temperature.
- Collisions are independent, identically distributed (i.i.d.) displacements ξ_n

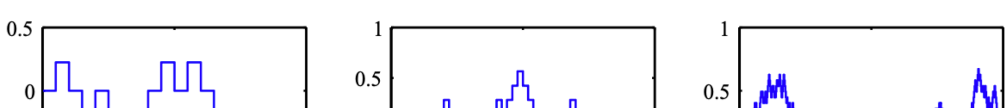
So lets fix the time interval $t\in[0,1]$, and assume there is <u>N collisions</u>, so that:

(particle position at time t) $$\boxed{X_t(N) = X_0 + \sum_{n=1}^{\lfloor Nt\rfloor} \xi_n, \text{ for } t\in[0,1]}$$

Then
$$\gamma = \mathbb{E}(X_1(N)-X_0)^2 = \mathrm{var}\left(\sum_{n=1}^{N}\xi_n\right) = N\,\mathrm{var}(\xi_n)$$

which implies an important information about.
$$\boxed{\mathrm{Var}(\xi_n) = \gamma N^{-1}}$$

– How does this process looks like as $N\to\infty$: [Figure from Handel '07]



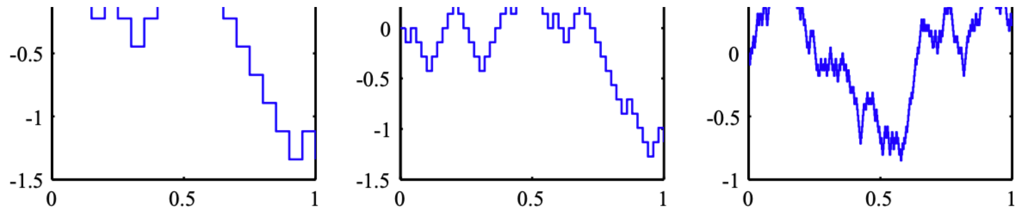


Figure 0.1. Randomly generated sample paths $x_t(N)$ for the Brownian motion model in the text, with (from left to right) $N = 20, 50, 500$ collisions per unit time. The displacements ξ_n are chosen to be random variables which take the values $\pm N^{-1/2}$ with equal probability.

lets rewrite this process in order to use the Central Limit Theorem:

$$X_t(N) = X_0 + \sqrt{\gamma t} \cdot \frac{\sum_{n=1}^{\lfloor Nt \rfloor} \Xi_n}{\sqrt{Nt}} \qquad \text{where } \Xi_n := \xi_n \cdot \sqrt{N\gamma^{-1}}$$

↑ i.i.d, zero-mean, unit variance

For fixed t, by CLT: as $N \to \infty$, $X_t(N) \xrightarrow{d} \mathcal{N}(0, \gamma t)$

Question: As $N \to \infty$, does the limit of the stochastic process $t \mapsto X_t(N)$ exist (in a suitable sense)?

[by Wiener] Yes, it is the "Brownian Motion", $\{W_t : t \in [0,1]\}$.
Wiener process

But it takes some serious effort to establish this!

Applications of the Wiener process W_t:

- Stochastic Control:

$$\min_{\{u_t\}} \mathbb{E}\left[\int_0^1 (W_t + Z_t)^2 dt\right] + \mathbb{E}\left[\int_0^1 u_t^2 dt\right]$$

$$\text{s.t. } \frac{dZ_t}{dt} = u_t$$

- Stock market ("Geometric Brownian Motion"):

(stock price) $S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$

(a unit stock price)
at time t

return rate

volatility

$\Rightarrow \mathbb{E}[S_t] = S_0 e^{\mu t}$ (why?).

- Estimation: measurement at continuous time t

"$y_t = x_t + \zeta_t$" $\quad$ x_t is positions of a particle, ζ_t is a "white noise".

integrated measurement signal $Y_t = \int_0^t y_s\,ds = \int_0^t x_s\,ds + W_t$

Find the best MSE estimate $\hat{x}_t$ of x_t s.t.

$$\{\hat{x}_s\}_0^1 = \underset{\{\bar{x}_s\}_0^1}{\arg\min}\ \mathbb{E}\left[\int_0^1 (y_s - \bar{x}_s)^2 ds\right]$$

We have a <u>serious issue</u> with W_t!

Suppose $\quad X_t = X_0 + W_t$

what is the derivative of X_t, $\dfrac{dX_t}{dt}$?

It <u>does not</u> exist a.s. because, even though $\{W_t\}$ is a well-defined process, <u>it is not differentiable a.e.</u>!

But we still like to make sense of "dW_t" s.t.

$$dX_t = dW_t \quad \equiv \quad X_t = \int_0^t dX_s = \int_0^t dW_t = W_t$$

More generally, we write

$$dX_t = b(X_t, t)\,dt + \sigma\,dW_t$$

to mean that

$$X_t = \int_0^t b(x_s, s)\, ds + \sigma W_t$$

and we can show that this is a unique solution by Picard iterations

$$\varphi_0(t) = X_0, \quad \varphi_{k+1}(t) = X_0 + \int_0^t b(\varphi_k(s), s)\, ds + \sigma W_t$$

and use Banach Fixed-Point theorem to argue that

$$\varphi_k(\cdot) \longrightarrow X_{(\cdot)} \text{ as } k \to \infty.$$

However, we still have some non-trivial challenges:

what $\frac{d}{dt} X_t^2 = ?!$

(naively) $\frac{d}{dt} X_t^2 = 2X_t \frac{dX_t}{dt} = 2X_t b(X_t, t) + 2\sigma X_t \, dW_t$

but now what does $\int_0^t X_t \, dW_t$ really mean ?!

This means we need some new calculus!

Most generally, we define the stochastic Differential Eq.

$$dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

by defining a new type of integral, called Itô Integral that can make sense of

" $\int_0^t \sigma(x_s, s)\, dW_s$ "

Also, it result in a "new chain rule":

for any $f \in C^2$, $df(W_t) = \underbrace{f'(W_t)\, dW_t}_{\text{usual term}} + \underbrace{\tfrac{1}{2} f''(W_t)\, dt}_{\text{2nd-order correction}}$

Using this formula, we can double check the variance of W_t as we designed to t:

$$
\begin{aligned}
Var(W_t) = \mathbb{E}(W_t^2) &= \mathbb{E}\left[\int_0^t \frac{d}{ds} W_s^2 \right] \\
&= \mathbb{E}\left[\int_0^t [2 W_s\, dW_s + ds] \right] \\
&= \mathbb{E}\left[\underbrace{2\int_0^t W_s\, dW_s}_{\text{zero-mean process (in fact a "martingale").}} + \int_0^t ds \right] = t
\end{aligned}
$$