Lecture 9

Monday, April 27, 2026 10:47 PM

Review:

Non-Integral Constraints Variational Problem (NIC-VP):

$$\min \; J(y) = \int_a^b L(x, y, y')\,dx$$

$\begin{cases} M: \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R} \\ M = M(x,y,z) \end{cases}$

s.t. $y \in C^1$, $y(a) = y_a$, $y(b) = y_b$

and $M(x, y(x), y'(x)) \equiv 0$, $\forall\, x \in [a,b]$.

→ we consider the augmented Lagrangian: $L + \lambda(x)\cdot M$

→ 1st-order Nec. condition for a regular curve y^* of $M \equiv 0$ on $[a,b]$:

$\exists\, \lambda^*(x): [a,b] \to \mathbb{R}$ s.t. y^* is an extremal for the augmented Lagrangian $L + \lambda(x)\cdot M$, i.e,

$$\left(L + \lambda^*(x)\cdot M\right)_y = \frac{d}{dx}\left(L + \lambda^*(x)\cdot M\right)_z .$$

→ Recall that y is regular curve of $M \equiv 0$ on $[a,b]$

if $M_z \neq 0$ $\forall\, x \in [a,b]$, or

when M is not a func. of y', $M_y \neq 0$ $\forall\, x \in [a,b]$.

→ special cases of $M \equiv 0$ on $[a,b]$:

1) $y' = f(x, y, u)$, u is free variable.

2) Holonomic constraints: (when M is not a func. of y', or it can be eliminated by taking an integration).

$\rightarrow$ 2nd-order nec. cond. for Basic Variational Problem:

$\delta J|_y(\eta) = 0$ and $\delta^2 J|_y(\eta) \geq 0 \quad \forall \eta$ admissible

But, $\delta^2 J|_y(\eta) = \int_a^b \left[(\eta')^T P(\eta') + (\eta^T) Q(\eta)\right] dx$

w/ $P(x) = \frac{1}{2} L_{zz}$ and $Q(x) = \frac{1}{2}\left(L_{yy} - \frac{d}{dx} L_{zy}\right)$

$\rightarrow$ Lemma: If $\int_a^b \left[(\eta')^T P(\eta') + (\eta^T) Q(\eta)\right] dx \geq 0 \quad \forall \eta \in C^1$, $\eta(a) = \eta(b) = 0$

then $P(x) \succeq 0$ P.S.D. $\forall x \in [a,b]$.

$\rightarrow$ Summary: 2nd-order necessary cond. for Basic VP

- $L_y = \frac{d}{dx} L_z$, $y(a) = y_a$, $y(b) = y_b$, and

- $\underbrace{L_{zz}(x, y(x), y'(x)) \succeq 0}_{\text{this is called Legendre's condition (1786).}}$ P.SD, $\forall x \in [a,b]$.

Recall the Hamiltonian $\rightsquigarrow H = H(x, y, z, p)$

$$H(x, y, y', p) = p \cdot y' - L(x, y, y')$$

$$\Rightarrow H_{zz} = -L_{zz}$$

For an extremal y, we defined $H^*(z) \triangleq H(x, y(x), z, p(x))$

and the 2nd-order nec. cond. means

that

$$\frac{d^2}{dz^2} H^*(z)\Big|_{z=y'(x)} = -L_{zz} \preccurlyeq 0$$ Negative semidefinite.

$\Rightarrow$ $\begin{cases} z = y'(x) \text{ is indeed a } \underline{\text{maximizer of the Hamiltonian}}. \\ H_{zz} \preccurlyeq 0 \quad \text{NSD}, \quad (\text{Legandre Cond.}) \\ \begin{cases} y' = H_p \\ p' = \frac{d}{dx} L_z = L_y = -H_y \end{cases} \quad (\text{Euler-Lagrange}) \end{cases}$

<u>Sufficient condition for a <u>weak</u> minimum</u>:

$\rightarrow$ from Taylor expansion of $J(y+\alpha\eta)$:

$$J(y+\alpha\eta) = J(y) + \delta J|_y(\eta)\cdot\alpha + \delta^2 J|_y(\eta)\cdot\alpha^2 + o(\alpha^2)$$

$\rightarrow$ you showed that (as an exercise of apply Taylor w/ Integral Remainder

$$o(\alpha^2) = \alpha^2 \int_a^b \left[(\eta'(x))^T \bar{P}(x,y,y',\alpha)(\eta'(x)) + (\eta(x))^T \bar{Q}(x,y,y',\alpha)\,\eta(x) \right] dx$$

where both $\bar{P}$ and $\bar{Q} \longrightarrow 0$ as $\alpha \to 0$ uniformly (~~tota~~) η

with respect to $\|\cdot\|_1$ norm: formally,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \ni \begin{cases} |\bar{P}(x,y,y';\alpha)| < \varepsilon \\ |\bar{Q}(x,y,y',\alpha)| < \varepsilon \end{cases} \quad \forall x \in [a,b],\ \forall \|\eta\|_1 \leq 1 \text{ and } \forall |\alpha| < \delta.$$

This ε and δ are <u>independent of the choice of perturbation η</u>.

Using this fact, our identical argument for sufficient cond. of optimality for $g(\alpha)$ would go through and we obtain that:

$$\boxed{\delta J|_y(\eta) = 0, \text{ and } \quad \delta^2 J|_y(\eta) > 0 \;\; \forall \eta \in C^1 \text{ admissible} \quad \text{implies } y \text{ is a } \underline{\text{strict}}\ \underline{\text{weak}} \text{ local minimum.}}$$

Can we simply this ?

It is very tempting to say that this reduces to

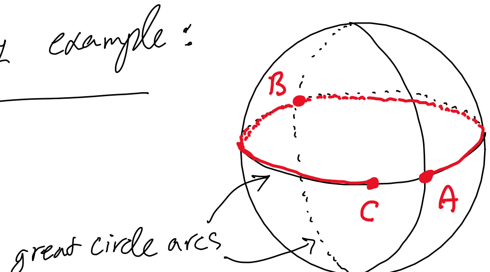
$$\delta^2 J|_y(\eta) = \int_a^b \left[(\eta')^T P(\eta') + (\eta^T) Q(\eta) \right] dx > 0, \quad \forall \eta \in C^1 \text{ admissible}$$

which we have argued similarly that reduces to

$$P = L_{zz} > 0 \quad \text{P.S.D. on } [a,b].$$

But, there is something <u>missing</u>! This condition is only capturing local properties of y w.r.t. the Lagrangian L. On the other hand, we know <u>locally minimizing curves are not necessarily globally minimizing</u>.

<u>Proof by example:</u>



- A–B is arc-length minimizing
- B–C " " "
- A–B–C is <u>not</u> arc-length minimizing

→ B is call a <u>conjugat point</u> of arc-length Lagrangian to point A.

observations: → There should be some additional global conditions for sufficiency.

→ This should not be needed if we consider the variational problem on a small enough region $[a,b]$ and small perturbations η.

. Both of these observations are true in general!

Lets try to formalize them in the scalar case: (Legendre's argument 1786)

$$\delta^2 J|_y(\eta) = \int_a^b [P.(\eta')^2 + Q.\eta^2]\,dx \geq 0 \quad \forall \eta \in C^1 \text{ admissible}$$

notice, for an arbitrary function $w = w(x)$: η admissible

$$\Rightarrow 0 = w.\eta^2\Big]_a^b = \int_a^b \frac{d}{dx}(w\eta^2)\,dx = \int_a^b (w'.\eta^2 + 2w.\eta.\eta')\,dx$$

$$\Rightarrow \delta^2 J|_y(\eta) = \int_a^b [P.(\eta')^2 + Q.\eta^2 + w'.\eta^2 + 2w.\eta.\eta']\,dx$$

$$= \int_a^b [P.(\eta')^2 + 2w.\eta'.\eta + (Q+w')\eta^2]\,dx$$

Now, if we choose w s.t.

$P(Q+w') = w^2, \quad \forall x \in [a,b]$ ← Riccati-type Diff. eq.

then, we have (already requiring that $P>0$ on $[a,b]$)

$$\delta^2 J|_y(\eta) = \int_a^b \left(\sqrt{P}.\eta' + \frac{w}{\sqrt{P}}.\eta\right)^2 dx = \int_a^b P(x)\left(\eta'(x) + \frac{w(x)}{P(x)}\eta(x)\right)^2 dx$$

∴ integral is always non-negative.

Now, because $P>0$, this ...

moreover, $\delta^2 J|_y(\eta) = 0$ implies that

$$\left.\begin{array}{l}\eta' + \frac{W}{P}\eta \equiv 0 \quad \text{on } [a,b] \\ \text{with boundary conditions} : \eta(a) = \eta(b) = 0\end{array}\right\} \Rightarrow \eta \equiv 0 \text{ on } [a,b].$$

Therefore, $\delta^2 J|_y(\eta) > 0$ if $\eta \not\equiv 0$ on $[a,b]$, and we can argue that y is a strict weak local minimum, as long as such $W = W(x)$ exists on $[a,b]$ satisfying

$$P(Q + W') = W^2 . \quad \leftarrow \text{Nonlinear Diff. eq.}$$

Question: Does this alway exist? No! example:

$P \equiv 1$, $Q \equiv -1$ on $[a,b]$ $\Rightarrow$ $W' = W^2 + 1$

$\Rightarrow$ $W = \tan(x - c)$ for some constant c.

$\Rightarrow$ $W(x)$ blows up if $(b-a) \geq \pi$

→ Existence of this solution implies that there is no conjugate point of Lagrangian on $[a,b]$, which is always true for small enough intervals $[a,b]$.

(This is Jacobi's argument 1837, 50 years later!)

→ Read pages 64–66.

. . -

<u>Thm:</u> (second order sufficient cond.)

An extremal $y(\cdot)$ is a strict weak minimum if

$L_{zz}(x, y, y') > 0$, $\forall x \in [a, b]$ <u>and</u> the interval $[a, b]$

contains no points conjugate to a.

<u>Note:</u> • this argument <u>will fail for</u> 0-norm $\|\eta\|_0$, so

our sufficient condition is only for <u>weak</u> minimum.

• For the vector case, everything is similar except that

$P(x) = \frac{1}{2} L_{zz} > 0$ P.D. on $[a, b]$
<u>Positive definite</u> matrix

and the Ricatti-type equation reads

$$\boxed{Q + W' = W P^{-1} W}$$