Lecture 8

Wednesday, April 22, 2026 8:23 PM

<u>Review:</u> Integral constrainted variational Problem (IC-VP)

$$\min \ J(y) = \int_a^b L(x, y, y')\, dx$$

s.t. $y \in C^1$, $y(a) = y_a$, $y(b) = y_b$ and

$$C(y) = \int_a^b M(x, y, y')\, dx = C_0 \qquad \begin{cases} M: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ M = M(x, y, z) \end{cases}$$

admissible perturbations η:

$\eta(a) = \eta(b) = 0$ and $C(y + \alpha\eta) = C_0 = C(y)$ for small enough α.

$$\Rightarrow \ \delta C|_y(\eta) = 0$$

$$\Rightarrow \int_a^b \left(M_y - \frac{d}{dx} M_z\right) \cdot \eta(x)\, dx = 0 .$$

- <u>1st-order necessary condition for IC-VP:</u>

$$\int_a^b \left(L_y - \frac{d}{dx} L_z\right) \cdot \eta(x)\, dx = 0, \ \forall \eta \in \left\{ \eta \in C^1 : \int_a^b \left(M_y - \frac{d}{dx} M_z\right) \cdot \eta(x)\, dx = 0 \text{ and } \eta(a) = \eta(b) = 0 \right\}.$$

- Using the idea of <u>regular curves for $C(y)$</u>, we can rewrite this

1st-order Nec. Cond.: If y is an extremal of the constrained problem, and y is <u>not</u> an extremal of $C(y)$, Then, $\exists \lambda^* \in \mathbb{R}$ constant such that y is an extremal of

$$(J + \lambda^* C)(y) = \int_a^b \left[L(x,y,y') + \lambda^* M(x,y,y') \right] dx$$

Non-Integral Constraints Variational Problem (NIC-VP):

$$\min \; J(y) = \int_a^b L(x,y,y')\, dx \qquad \begin{cases} M: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ M = M(x,y,z) \end{cases}$$

s.t. $y \in C^1$, $y(a) = y_a$, $y(b) = y_b$

and $M(x, y(x), y'(x)) = 0$, $\forall\, x \in [a,b]$.

- This is infinitely many constraints!

but consider a single $x_0 \in [a,b]$:

delta-dirac!

$$0 = M(x_0, y(x_0), y'(x_0)) = \int_a^b M(x, y(x), y'(x))\, \delta_{x_0}(x)\, dx.$$

So from the 1-st order nec. cond. of IC-VP

the augmented Lagrangian for x_0 would take the form

$$L + \lambda^*_{x_0} \cdot \delta_{x_0}(x) \cdot M(x, y(x), y'(x)), \qquad \lambda^*_{x_0} \in \mathbb{R}. \text{ Constant!}$$

Q: How to generalize this for all $x \in [a,b]$?

we have this augmentation for all $x_0 \in [a,b]$, so the augmented Lagrangian the whole Non-integral constaints $\forall x \in [a,b]$ is

$$L + \int_a^b \lambda^*_{x_0}\, \delta_{x_0}(x) \cdot M(x, y(x), y'(x))\, dx_0$$

$$= L + \lambda^*(x) \cdot M(x, y(x), y'(x))$$

where $\overset{*}{\lambda}(x) \triangleq \int_a^b \lambda_{x_0} \delta_{x_0}(x)\, dx_0$ is a function on $[a,b]$,

called the "distributed Lagrange multiplier".

So, we can consider minimizing the unconstrained problem

$$\min \int_a^b [\mathcal{L}(x,y,y') + \overset{*}{\lambda}(x) \cdot M(x,y,y')]\, dx .$$

→ what about regularity conditions for y ?!

Def: y is a regular curve for Non-integral constraint $M \equiv 0$ on $[a,b]$

if $M_z \neq 0$ on $[a,b]$, or when M does not depend on y',

if $M_y \neq 0$ on $[a,b]$.

Similarly, this Regularity of y allows us to use Implicit func. theorem.

special cases of Non-Integral Constraint : $M(x,y,y') \equiv 0$ on $[a,b]$

→ In general, M depends on y' but is an underdetermined system

of equations, so we can solve for y' as a function of x, y,

and some free variables u :

$$y' = f(x,y,u)$$

example: $y \in C^2([a,b], \mathbb{R}^2)$ and $M(x,y,y') = y_1' - y_2$

then $y_1' = y_2$ and $y_2' = u$ (free), $u \in C([a,b], \mathbb{R})$;

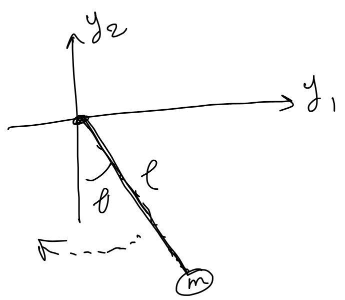
or $\quad y' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underset{\text{state}}{y} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underset{\text{control}}{u} \quad \rightsquigarrow$ a "control system"

<u>Note:</u> The unconstrained variational problem could be thought as
$y' = u$ (free) with $u \in C([a,b], \mathbb{R}^n)$.

→ "Holonomic constraints": when m does not depend on y'.
or, m depends on y' but it can be eliminated <u>by an integration</u>!

<u>example:</u> Find equations of motions for Pendulum:

$x \rightsquigarrow$ "t" time



Constraint:

$$m(t,y) = y_1^2 + y_2^2 - \ell^2 = 0, \ \forall t.$$

$\rightsquigarrow$ holonomic!

$$\min \int_{t_0}^{t_1} (T - U)\, dt \quad \text{s.t. } m(t,y) = 0 \quad \forall t \in [t_0, t_1]$$

<u>Exercise:</u> Solve this holonomic constrained variational problem, first in (y_1, y_2) coordinates, then with a change of coordinates to (r, θ) polar.

Hint: in (r,θ), $m(t,y) = 0$ reduces to $r \equiv \ell, \forall t$
and <u>θ is free</u>.

Finally, in optimal control, we are interested in Non-holonomic constraints

(which cannot be integrated to omit y') ; i.e.,

two points on y-space can be connected by an admissible curve satisfying the constraints, and there is No lower-dimensional constraints surface !

Second-order Necessary Conditions:

We consider the Basic variational problem :

Recall that $J(y+\alpha\eta) = J(y) + \delta J|_y(\eta)\cdot\alpha + \delta^2 J|_y(\eta)\cdot\alpha^2 + o(\alpha^2)$

1st-order nec. cond. $\rightsquigarrow$ $\delta J|_y(\eta) = 0$

2nd-order nec. cond $\rightsquigarrow$

$$\delta J|_y(\eta) = 0 \quad \text{and} \quad \delta^2 J|_y(\eta) \geq 0 \quad , \forall \eta \text{ admissible}$$

- lets compute the 2nd variation $\delta^2 J|_y(\eta)$: supp. $L \in C^3$,

$$J(y+\alpha\eta) = \int_a^b L(x, y+\alpha\eta, y'+\alpha\eta')\,dx$$

$$= \int_a^b L\,dx + \int_a^b \left(L_y\cdot\eta + L_z\cdot\eta'\right)\alpha\,dx$$

$$+ \int_a^b \frac{\alpha^2}{2}\left(\eta^T L_{yy}\eta + 2\eta^T L_{yz}\eta' + \eta'^T L_{zz}\eta'\right)dx + o(\alpha^2)$$

By Integration by parts :

$$\int_a^b 2\eta^T L_{yz}\eta'\,dx = -\int_a^b \eta^T\left(\frac{d}{dx}L_{yz}\right)\eta\,dx + \cancel{\eta^T L_{yz}\eta\Big]_a^b}\; 0$$

and for Basic VP $\rightsquigarrow$ $\eta(a)=\eta(b)=0$ $\Rightarrow$

$$\Rightarrow \delta^2 J\big|_y(\eta) = \int_a^b \tfrac{1}{2}\left[\eta^T\left(L_{yy}-\frac{d}{dx}L_{yz}\right)\eta + \eta'^T L_{zz}\eta'\right]dx$$

$$= \int_a^b \left[\eta_{(x)}^T Q(x)\,\eta_{(x)} + \eta'^T_{(x)} P_{(x)}\,\eta'_{(x)}\right]dx$$

where $P(x) \triangleq \frac{1}{2}L_{zz}$, $Q(x) \triangleq \frac{1}{2}\left(L_{yy}-\frac{d}{dx}L_{yz}\right)$ are both continuous when $L\in C^3$ and $y\in C^2$.

Lemma 3: (3rd Fundamental Lemma of Calculus of variations)

If $\int_a^b \left(\eta'^T P\eta' + \eta^T Q\eta\right)dx \geq 0$ $\forall\, \eta\in C^1$ s.t. $\eta(a)=\eta(b)=0$

Then, $P_{(x)} \succeq 0$ P.S.D. $\forall x\in[a,b]$.

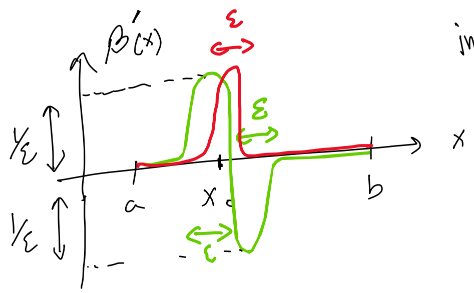
Proof: suppose at $x_0\in[a,b]$, $P(x_0)$ is not P.S.D.

But $P(x)$ is continuous so

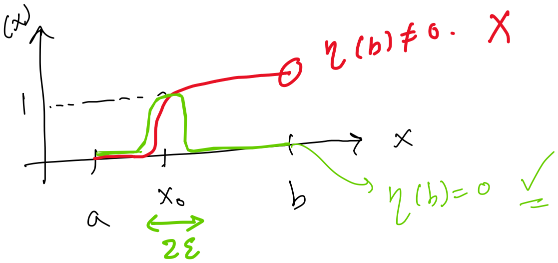
$\exists\, v\in\mathbb{R}^n$ and $\varepsilon,\delta>0$: s.t. $v^T P(x)v < -\delta < 0$, $\forall x\in[x_0-\varepsilon, x_0+\varepsilon]$

lets design a "bad η" s.t. the integral is negative!

1-dim visualization



integrate $\Rightarrow$

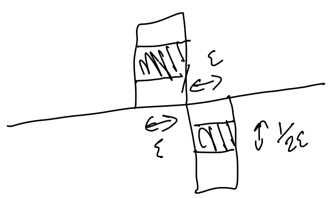


Lets pick the green curve as a multiplier and choose

$\eta_\varepsilon(x) = \beta(x)\, v^*$, and compute the integral:

$$\int_a^b \eta_\varepsilon'^T P \eta_\varepsilon' \, dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \underbrace{v^T P(x) v}\cdot [\beta'(x)]^2 \, dx$$

$$\le -\delta \cdot \int_{x_0-\varepsilon}^{x_0+\varepsilon} [\beta'(x)]^2 \, dx$$



$$\rightsquigarrow \quad \le -\delta \cdot 2\varepsilon \cdot \left(\frac{1}{2\varepsilon}\right)^2 = \frac{-\delta}{2\varepsilon}$$

$$\left| \int_a^b \eta_\varepsilon^T Q \eta_\varepsilon \, dx \right| = \left| \int_{x_0-\varepsilon}^{x_0+\varepsilon} \eta_\varepsilon^T Q \eta_\varepsilon \right| dx \le \int_{x_0-\varepsilon}^{x_0-\varepsilon} \left| \eta_\varepsilon^T Q \eta_\varepsilon \right| dx$$

$$\le \max_{x\in[a,b]} \left| v^T Q(x) v \right| \cdot \int_{x_0-\varepsilon}^{x_0+\varepsilon} \beta(x)^2 \, dx$$

$\le C \cdot 2\varepsilon$ for some constant C and

$C < \infty$ because $Q(x)$ is continuous on $[a,b]$.

Therefore, $\displaystyle\int_a^b (\eta'^T P \eta' + \eta^T Q \eta)\, dx \le \frac{-\delta}{2\varepsilon} + 2C\cdot\varepsilon$

and as $\varepsilon \to 0$, the right hand side goes to $-\infty$.

This is a contradiction, so $P(x) \succcurlyeq 0$ P.S.D on all $x \in [a,b]$. $\boxtimes$

$\Rightarrow$ 2nd-order necessary cond. for Basic VP:

$$\begin{cases} L_y = \dfrac{d}{dx} L_z , \quad y(a) = y_a , \; y(b) = y_b . \\ \text{and} \\ \underbrace{L_{zz}(x, y(x), y'(x)) \succcurlyeq 0}_{\text{this is called Legendre's condition (1786).}} \quad \text{P.S.D}, \quad \forall x \in [a,b]. \end{cases}$$

Recall the Hamiltonian $\rightsquigarrow H = H(x, y, z, P)$

$$H(x, y, y', P) = P \cdot y' - L(x, y, y')$$

$$\Rightarrow H_{zz} = - L_{zz}$$

For an extremal y, we defined $H^*(z) \triangleq H(x, y(x), z, P(x))$

and the 2nd-order nec. cond. means

$$\frac{d^2}{dz} H^*(z) \Big|_{z = y'(x)} = - L_{zz} \preccurlyeq 0 \quad \text{negative semidefinite.}$$

$\Rightarrow$ $z = y'(x)$ is indeed a <u>maximizer</u> of the Hamiltonian.

$\rightarrow$ Reading assignment Chap. 2.

$\rightarrow$ <u>Exercise</u>: Exercise 2.12 from the book.