Lecture 7

Monday, April 20, 2026 4:35 PM

Review: so far we considered

$$\min \ J(y) \triangleq \int_a^b L(x,y,y')\,dx$$

$$\text{s.t. } y \in C^1,\ y(a) = y_a,\ y(b) = y_b$$

Nec. Cond: Euler-Lagrange differential eq. $L_y = \frac{d}{dx} L_z$.

Boundary conditions:

1) $(a, y_a), (b, y_b)$ are fixed. ✓

2) y_b is free $\Rightarrow$ $L_z\big|_b = 0$ and $y(a) = y_a$.

3) b and y_b are free, but $y(b) = \varphi(b)$ for a given $\varphi \in C^1$ $\Rightarrow$ "you obtain it!"

Hamiltonian Canonical Form:

Hamiltonian $\rightsquigarrow$ $H(x, y, y', p) := p \cdot y' - L(x,y,y')$

moment vector $\rightsquigarrow$ $p(x) := L_z(x,y,y')$.

→ Then, E-L eq. translates to Hamiltonian canonical eq.

$$\boxed{y' = H_p \quad \text{and} \quad p' = -H_y}$$

→ And, if $H^*(z) := H(x, y, z, p)$ then at extremals y we have

$$\frac{d}{dz} H^*(z)\Big|_{z=y'(x)} = 0 \quad \text{and} \quad \frac{d^2}{dz^2} H^*(z)\Big|_{z=y'(x)} \geq 0.$$

i.e, $z = y'(x)$ is a maximum of Hamiltonian H^*.

Physical Interpretations:

- Motion of a particle with mass m in a potential field U, with kinnetic energy T.
- The particle follows "Hamilton's principle of least action", i.e. takes a path that minimizes $\int_{t_0}^{t_1} (T-U)\,dt$.
- The Hamiltonian represents the total energy $H = T + U$.

"Noether's Theorem":

- An invariance of $\underset{\text{action integral}}{\underline{\text{Lagrangian}}}$ under a $\underset{\text{time-shift, translation, ---}}{\underline{\text{transformation}}}$ implies the existence of a conserved quantity.

Constrained Variational problem:

I) Integral Constraints

$$C(y) \triangleq \int_a^b M(x, y(x), y'(x))\, dx = C_0 \rightarrow \text{given}$$

example: Dido's Isoperimetric Problem

II) Non-Integral Constraints

$$M(x, y(x), y'(x)) = 0 \qquad \forall\, x \in [a,b]$$

example: optimal control problems

Integral constraints:

$$\min \; J(y) = \int_a^b L(x, y, y')\, dx$$

s.t. $y \in C^1$, $y(a) = y_a$, $y(b) = y_b$ and

$$C(y) = \int_a^b M(x, y, y')\, dx = C_0 \qquad \begin{cases} M : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ M = M(x, y, z) \end{cases}$$

what are the admissible perturbations η?

$\eta(a) = \eta(b) = 0$ and $C(y + \alpha\eta) = C_0$ for small enough α.

this means: $C(y + \alpha\eta) = C(y) \quad \forall\, |\alpha| < \varepsilon$

$$\Rightarrow \delta C|_y(\eta) = 0$$

$$\Rightarrow \int_a^b \left(M_y - \frac{d}{dx} M_z\right) \cdot \eta(x)\, dx = 0. \qquad (**)$$

Therefore, the 1-st order necessary condition is

$$\int_a^b \left(L_y - \frac{d}{dx} L_z\right) \cdot \eta(x)\, dx = 0, \quad \forall\, \eta \in C^1 : \int_a^b \left(M_y - \frac{d}{dx} M_z\right) \cdot \eta(x)\, dx = 0$$

and $\eta(a) = \eta(b) = 0$

Let's, rewrite this nec. condition to understant it better.

Define the inner product $\langle \cdot, \cdot \rangle$ on $L^2([a,b], \mathbb{R}^n)$, square integrable functions from $[a,b]$ to $\mathbb{R}^n$, as:

function:

$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x) \, dx$$

Then, the nec. cond. reads:

$$\langle L_y - \frac{d}{dx} L_z, \eta \rangle = 0 \ , \ \forall \eta: \langle M_y - \frac{d}{dx} M_z, \eta \rangle = 0$$

Recall are argument from finite dimensional that, we can write this equivalently as:

$\exists$ constant $\lambda^* \in \mathbb{R}$ (Lagrange multiplyer) such that

$$\left(L_y - \frac{d}{dx} L_z\right) + \lambda^* \left(M_y - \frac{d}{dx} M_z\right) = 0 \ , \ \forall x \in [a,b].$$

Rearrange terms to argue that y is extremal of the "augmented Lagrangian" $\underline{L + \lambda^* M}$, because the corresponding E-L. eq. would be

$$L_y + \lambda^* M_y - \frac{d}{dx}\left(L_z + \lambda^* M_z\right) = 0 \ . \ \checkmark$$

Question: Is this argument completely correct? No!

I) the condition in (**) is only necessary for η to be

... don't know if this is

an admissible perturbation, but ...

also sufficient.

II) Recall that the finite-dimensional argument was only true if x^* was a regular point of D, i.e., $\{\nabla h_i(x^*)\}_{i=1}^m$ were linearly independent. Here, a Non-regular point means that y satisfies the constraint but all near by curves violates it; or equivalently, y is an extremal of functional $C(y)$.

Def: y is a regular curve for constraint $C(y)$ if $\nexists\, \lambda^* \in \mathbb{R}$

$$M_y(x,y,y') \neq \frac{d}{dx} M_z(x,y,y') \quad \text{for some } x \in [a,b].$$

$\rightsquigarrow$ more, generally y is a regular curve for $C(y)$ if there exist nearby curves with lower and higher values than $C(y)$.

1st order Nec. Cond.: If y is an extremal of the constrained problem, and y is not an extremal of $C(y)$; Then, $\exists\, \lambda^* \in \mathbb{R}$ constant such that y is an extremal of

$$(J + \lambda^* C)(y) = \int_a^b \left[L(x,y,y') + \lambda^* M(x,y,y') \right] dx$$

Proof: Choose a two-parameter family of perturbations $y+\alpha_1\eta_1+\alpha_2\eta_2$ and Use Inverse function theorem.

Exercise: Solve Dido's Isoperimetric and Catenary problems!

Multiple Integral constraints:

$$\min_{y \in C^1} J(y) \quad \text{s.t.} \quad C_i(y) = C_0^{(i)} \quad \text{for } i=1,\dots,m, \quad y(a)=y_a,\ y(b)=y_b.$$

If y is an extremal of this problem, and y is regular for all $C_i(y)$, there exists λ_i^*, $i=1,\dots,m$ such that y is extremal of the unconstrained augmented problem:

$$J(y) + \lambda_1^* C_1(y) + \lambda_2^* C_2(y) + \dots + \lambda_m C_m(y).$$