Lecture 6

Wednesday, April 15, 2026 5:47 PM

Review:

- Fixed end-point variational problem:

$$\min \ J(y) = \int_a^b L(x, y(x), y'(x))\, dx \quad , \ L \in C^1$$

$$\text{s.t.} \quad y \in C^1([a,b], \mathbb{R}^n) \ , \ y(a) = y_a \ , \ y(b) = y_b .$$

- Nec. Cond.: Euler-Lagrange equation

$$\boxed{L_y = \frac{d}{dx} L_z \quad \text{w/ BC}: \ y(a) = y_a, \ y(b) = y_b.}$$

- Smoothness Condition:

by changing the term for integration by part, we can obtain nec. cond as:

$$\int_a^b \left[\int_a^x L_y \, ds - L_z \right] \eta'(x)\, dx = 0, \ \forall \eta \text{ admissible.}$$

Then, by 2nd-Fund. lemma of Cal. of variations

$$\int_a^x L_y \, ds - L_z \equiv C \quad \text{on } [a,b].$$

$\Rightarrow$ if $y \in C^1$ is an extremum L_z must be continuously differentiable and we get

... applying $\frac{d}{\ }$ to both sides

EL. eq. back by applying $\frac{d}{dx}$

So, $y \in C^1$ and $\mathcal{L} \in C^1$

- <u>Variable end-point problem:</u>

suppose a, y_a, b are given but y_b is free!

Then, the nec. cond. becomes:

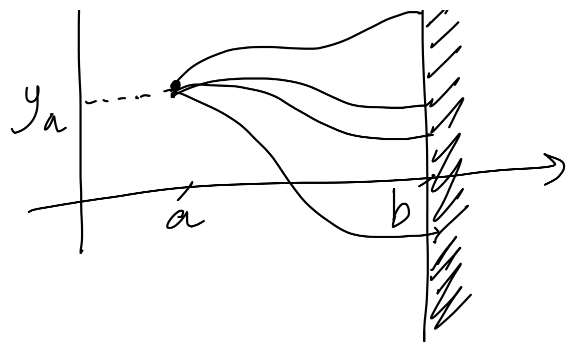
$$0 = \delta J\big|_y(\eta) = \int_a^b \left(\mathcal{L}_y - \frac{d}{dx}\mathcal{L}_z \right) \eta(x)\, dx + \mathcal{L}_z \cdot \eta \Big]_a^b$$

$$, \underbrace{\forall\, \eta \in C^1 \text{ s.t. } \eta(a) = 0 \text{ but } \eta(b) \text{ is free.}}_{\text{Admissible perturbations}}$$

Then, by sequentially interpreting this condition we get the Nec. Cond. for this case as:

$$\boxed{\mathcal{L}_y = \frac{d}{dx}\mathcal{L}_z \,, \quad y(a) = y_a \,, \quad \mathcal{L}_z\big|_b = 0}\,.$$

<u>Example:</u> Find distance minimizing curves from a point to a <u>vertical wall</u>:

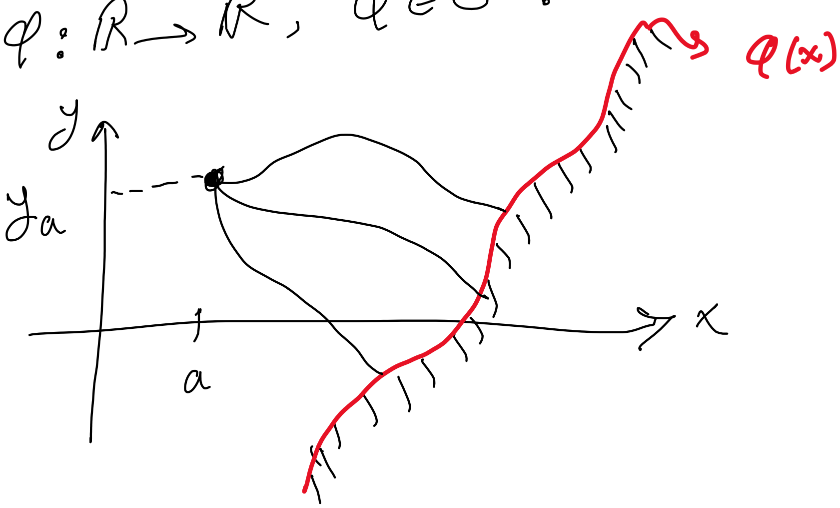


→ Solve this problem and interpret the condition

$L_z|_b = 0$!

<u>Exercise</u>: what if instead of a wall we are given a curve parameterized by

$\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \in C^1$.



Note that (a, y_a) is fixed, but both b and y_a are free and instead we have the condition $y(b) = \varphi(b)$!

Hamiltonian canonical equations :

Recall the momentum vector :

$$p(x) := L_z(x, y(x), y'(x))$$

and the Hamiltonian scalar :

inner-product!

$$H(x, y(x), y'(x), p(x)) := p(x) \cdot y'(x) - L(x, y(x), y'(x))$$

- Along a fixed curve y, $x \mapsto H(x, y, y', p)$ is a function of x alone!

- suppose y is an extremal $\Rightarrow L_y = \frac{d}{dx} L_z$.

Then,
$$\frac{dy}{dx} = y'(x) = \frac{\partial}{\partial p} H = H_p$$

and
$$\frac{dp}{dx} = \frac{d}{dx} L_z = L_y = -\frac{\partial}{\partial y} H = -H_y .$$

E.L. eq.

Therefore, for extremal y we have

$$\boxed{y' = H_p , \quad p' = -H_y}$$

"Hamiltonian's canonical eqs."

Note: since we are not in "no x" or "no y" special case, the momentum p and Hamiltonian H are not necessarily constant along extremals.

Exercise: what if we are in "no x" or "no y" special cases for L?

Another important observation:

$$\frac{\partial}{\partial y'}(H) = p - L_z(x, y, y') = 0$$

or more precisely: for any fixed $x \in [a,b]$

let $H^*(z) = H(x, y(x), z, p(x))$ where y is an extremal. Then

$$\frac{d}{dz} H^*(z) \Big|_{z = y'(x)} = 0$$

In fact, we can also show that $z = y'(x)$ is a maximum of the Hamiltonian.

→ this is the basis of Pontryagin Maximum principle we will develop later!

Physical Intuitions / Interpretations

- The principle of least action and conservation laws

Recall Newton's second law of motion:

t : time

q : coordinate vector

$U(q)$: potential energy

$$\frac{d}{dt}(m \dot{q}) = -U_q \leftarrow \text{"Conservative force"}$$

..

Now, if we define the Lagrangian

$\mathcal{L} \triangleq T - U$ = kinetic - potential

$\mathcal{L}(t, q, \dot{q}) = \frac{1}{2} m (\dot{q} \cdot \dot{q}) - U(q)$

and consider the variational problem:

$$\min_{q \in C^1} \int_{t_0}^{t_1} \mathcal{L} \, dt = \int_{t_0}^{t_1} (T - U) \, dt$$

Then, when $t_1 - t_0$ is small enough, the extremal q^* indeed minimizes this functional, called

"Hamilton's principle of least action"

→ what is Euler-Lagrange equation?

$$\mathcal{L}_q = \frac{d}{dt}(\mathcal{L}_{\dot{q}}) \Rightarrow -U_q = \frac{d}{dt}(m \dot{q})$$

So, we get Newton's second law!

→ what is the Hamiltonian ?

$$H(t, q, \dot{q}, p) = p \cdot \dot{q} - L$$

$$= L_{\dot{q}} \cdot \dot{q} - L$$

$$= (m\dot{q}) \cdot \dot{q} - \frac{1}{2} m \dot{q} \cdot \dot{q} + U(q)$$

$$= \frac{1}{2} m \dot{q} \cdot \dot{q} + U(q)$$

$$= T + U = E$$ total energy !

Note: If $U \equiv 0$ no potential energy, then the Lagrangian is only a function of $\dot{q}$ and thus the extremals are "straight lines".
If $U \not\equiv 0$, then the extremals will be straight lines (geodesics) in a curved space whose metric is determined by gravitational forces. ⇝ this is the basis of Einstein's theory of general relativity.

Exercise: Consider the "no x" and "no y" special cases of Lagrangian and Interpret their physics!

1) "no x" means "no t" here $\Rightarrow \mathcal{L} = \mathcal{L}(q, \dot{q})$

$$\Rightarrow \frac{d}{dt}\left(\mathcal{L}_{\dot{q}} \cdot \dot{q} - \mathcal{L}\right) = 0 \Rightarrow H \equiv C \text{ on } [t_0, t_1] \text{ Constant.}$$

Hamiltonian is conserved along extremal curves

$\Rightarrow$ "Conservation of Energy"

2) "no y" means "no q" here $\Rightarrow \mathcal{L} = \mathcal{L}(t, \dot{q})$

$$\frac{d}{dt}(\mathcal{L}_{\dot{q}}) = \mathcal{L}_q \equiv 0 \text{ on } [t_0, t_1] \Rightarrow \mathcal{L}_{\dot{q}} \equiv C \text{ on } [t_0, t_1]$$

$\Rightarrow$ $m\dot{q}$ is conserved along extremal curves

$\Rightarrow$ "Conservation of Momentum"

A General theorem: "Noether's theorem"

An invariance of action integral under a transformation (e.g. time shift, translation, rotation, ...) implies the existence of a conserved quantity.