Lecture 5

Thursday, April 9, 2026 11:37 AM

<u>Review</u>: Basic variational problem $\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ <u>running cost</u> Lagrangian

$$\min \ J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

$$\text{s.t. } y \in C^1([a,b], \mathbb{R}^n), \quad y(a) = y_a, \quad y(b) = y_b.$$

The 1st-order nec. cond. "$\delta J|_y(\eta) = 0 \ \forall$ admissible η" reduces to Euler-Lagrange Equation if $\mathcal{L}$ and y are piece-wise C^1:

$$\boxed{\mathcal{L}_y = \frac{d}{dx} \mathcal{L}_z}$$

and for $n > 1$, it is interpreted as

$$\mathcal{L}_{y_i}(x, y(x), y'(x)) = \frac{d}{dx} \mathcal{L}_{z_i}(x, y(x), y'(x)) \quad \text{for } i = 1, 2, \dots, n.$$

with Boundary Cond: $y(a) = y_a$, $y(b) = y_b$.

- <u>Two special cases</u>:

1) "no y" dependence: $0 = \mathcal{L}_y = \frac{d}{dx} \mathcal{L}_z \Rightarrow \mathcal{L}_z \equiv C$ constant!

called momentum

2) "no x" dependence: $\mathcal{L}_y = \frac{d}{dx} \mathcal{L}_z \rightsquigarrow \mathcal{L}_y \cdot y' = \mathcal{L}_z (y')^2 + \mathcal{L}_z y'' \cdot y'$

$$\Rightarrow \frac{d}{dx}\left(\mathcal{L}_z \cdot y' - \mathcal{L}\right) = 0 \Rightarrow \underbrace{\mathcal{L}_z \cdot y' - \mathcal{L}}_{\text{called Hamiltonian}} \equiv C \text{ constant!}$$

lets explain case 2: $\mathcal{L}$ has "no x" direct dependence

$$\mathcal{L} = \mathcal{L}(y, y') \Rightarrow \mathcal{L}_y = \mathcal{L}_{zy} \cdot y' + \mathcal{L}_{zz} \cdot y''$$

(multiply by y') $\Rightarrow$ $\mathcal{L}_y \cdot y' = \mathcal{L}_{zy}(y')^2 + \mathcal{L}_{zz} \cdot y'' \cdot y'$

$$\Rightarrow \quad \mathcal{L}_{zy}(y')^2 + \mathcal{L}_{zz} \cdot y'' \cdot y' - \mathcal{L}_y \cdot y' = 0$$

Compute $\frac{d}{dx}(\mathcal{L}_z \cdot y' - \mathcal{L}) =$

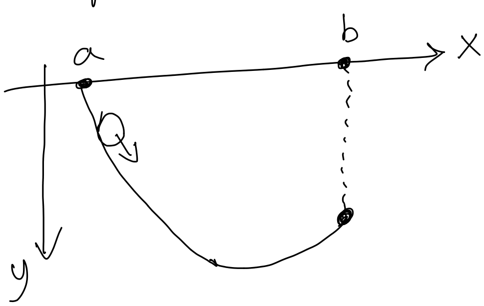
$$\cancel{\mathcal{L}_{zx} \cdot y'} + \mathcal{L}_{zy} \cdot y' \cdot y' + \mathcal{L}_{zz} \cdot y'' \cdot y' + \underline{\mathcal{L}_z y''} - \cancel{\mathcal{L}_x} - \mathcal{L}_y y' - \underline{\mathcal{L}_z \cdot y''}$$

$$= \mathcal{L}_{zy} \cdot (y')^2 + \mathcal{L}_{zz} y'' \cdot y' - \mathcal{L}_y \cdot y' = 0$$

$$\Rightarrow \boxed{\begin{array}{c} \mathcal{L}_z(x, y(x), y'(x)) \cdot y'(x) - \mathcal{L}(x, y(x), y'(x)) \equiv C \\ \text{for any constant } C \in \mathbb{R} \end{array}}$$

"$\mathcal{L}_z \cdot y' - \mathcal{L}$" is called the "Hamiltonian"

Example: Brachistochrone (shortest time) problem:



- Given fixed a, b, $y(a) = 0$, $y(b) = y_b$.
- Consider sliding a ball with no friction, no initial velocity down a curve slide.
- Find the profile of slide that minimizes the time of reach to (b, y_b).

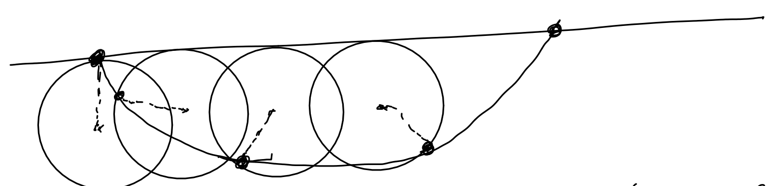
→ kinetic + potential energy $\equiv 0$. $\Rightarrow \frac{mv^2}{2} - mgy = 0$

⇝ up to constant: $v = \sqrt{y}$.

⇝ time to travel ds: $\frac{ds}{v} = \frac{\sqrt{1+(y'(x))^2}\cdot dx}{\sqrt{y(x)}}$.

$$\underset{\substack{y\in C^1([a,b]) \\ y(a)=0,\ y(b)=y_b}}{\text{Min}} \int_a^b \frac{\sqrt{1+(y'(x))^2}}{\sqrt{y(x)}} \cdot dx$$

→ sol: cycloids: $\begin{cases} x(\theta) = a + c(\theta - \sin\theta) \\ y(\theta) = c(1-\cos\theta). \end{cases}$ $\theta\in[0, 2\pi]$.



roll a circle without slipping (radius c).

Exercise: use the "no x" case to show this is the optimal solution.

Smoothness requirement of y and $\mathcal{L}$ to obtain E.L. eq.

Recall, we obtained EL eq. by "1st Fundamental lemma of cal. of variations":

Lemma: supp. $\xi : [a,b] \to \mathbb{R}$ is cont..

If $\int_a^b \xi(x) \cdot \eta(x) = 0 \quad \forall \eta \in C'([a,b]) \ni \eta(a) = \eta(b) = 0$

then $\xi(x) \equiv 0$ on $[a,b]$.

and removing η' from the 1st-order nec. cond.

$$0 = \delta J|_y(\eta) = \int_a^b (L_y \cdot \eta + L_z \cdot \eta')\, dx .$$

But, then

$$L_y = \frac{d}{dx} L_z = L_{zx} + L_{zy} \cdot y' + L_{zz} \cdot y''$$

so it seems like we need to assume that both $\mathcal{L}$ and y are C^2, rather than C'.

But, if we update our argument we see that C' smoothness assumption suffices and in fact we recover C^2 smoothness as a result; i.e., the extremal curves y will automatically enjoy more C^2-smoothness!

Lets update our argument : Recall

$$\delta J|_y(\eta) = \int_a^b [L_y(x,y,y')\eta + L_z(x,y,y')\eta'] \, dx$$

(Integration by part for $L_y \cdot \eta$)

$$= -\int_a^b \left(\int_a^x L_y(s, y(s), y'(s)) \, ds \right) \cdot \eta'(x) \, dx + \left(\int_a^x L_y \, ds \right) \eta(x) \Big]_a^b + \int_a^b L_z \cdot \eta'(x) \, dx$$

(why ?)

$$= \int_a^b \left[L_z(x, y(x), y'(x)) - \int_a^x L_y(s, y(s), y'(s)) ds \right] \eta'(x) \, dx$$

1.st-order nec. Cond : $\delta J|_y(\eta) = 0 \quad \forall \eta$ admissible.

Lemma 2 : (2nd Fundamental Lemma of Calc. of variations)

Supp. $\int_a^b \xi(x) \eta'(x) \, dx = 0 \quad \forall \eta \in C^1([a,b])$, $\eta(a) = \eta(b) = 0$

If $\xi : [a,b] \to \mathbb{R}$ is continuous, then ξ is a constant func.

Proof: Exercise !

Applying this Lemma implies that :

$$L_z(x, y(x), y'(x)) = \int_a^x L_y(s, y(s), y'(s)) \, ds + \underset{\text{Constant !}}{C}$$

$\Rightarrow$ if L_y is continuous then L_z is C^1 and $\frac{d}{dx} L_z = L_y$.

So we don't need any assumption more that $\eta, L \in C^1$.

Generalization: our analysis can be similarly generalized to show that the E.L. eq. even applies under piece-wise C^1 assumption. (continuous curves on $[a,b]$ w/ continuous derivative everywhere except finitely many points).

Exercise: How? Generalize Lemma 2!

Variable end-point variational problem:

Recall that the basic variational problem had the boundary conditions $y(a)=y_a$, $y(b)=y_b$.

This implied that $\eta \in C^1$ is admissible iff $\eta(a)=\eta(b)=0$. $\Rightarrow$ E.L. equations.

Q: what if $y(b)$ was free?

In this case, $\eta \in C^1$ is admissible iff $\eta(a)=0$ and $\eta(b) \in \mathbb{R}$ is free variable.

Recall that we can similarly obtain that

$$\delta J|_y(\eta) = \int_a^b \left(L_y - \frac{d}{dx} L_z \right) \eta(x)\, dx + L_z \cdot \eta(x) \Big]_a^b$$

$$= \int_a^b \left(L_y - \frac{d}{dx} L_z \right) \eta(x)\, dx + L_z \cdot \eta(b)$$

(Nec. Cond.) $= 0 \quad \forall\, \eta$ admissible.

$\rightarrow$ this includes $\eta \in C^1$ s.t. $\eta(a) = \eta(b) = 0$

$$\Rightarrow \int_a^b \left(L_y - \frac{d}{dx} L_z \right) \eta(x)\, dx = 0 \quad \forall \text{ such } \eta.$$

So, by fundamental lemma of calc. of variations:

$$\Rightarrow \boxed{L_y - \frac{d}{dx} L_z \equiv 0 \quad \text{on } [a,b].}$$

But, then back to (Nec. Cond) above, we have also

$$L_z \cdot \eta(b) = 0 \quad \forall\, \eta \text{ admissible}$$

and $\eta(b)$ is arbitrary. So,

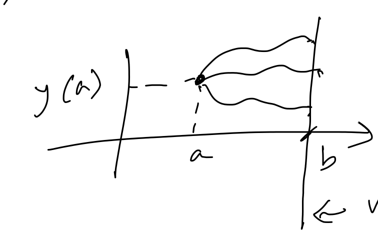
$$\boxed{L_z|_b = L_z(b, y(b), y'(b)) = 0}$$

this is a new condition replacing $y(b)=y_b$ appearing because $y(b)$ is free !

Exercise: Find the shortest path from a point, $(a, y(a)) \in \mathbb{R}^2$ to a vertical line at $b \neq a$.

(a, y_a, b are fixed)

point:



Question: what if instead of the given vertical line we are given a curve given by $\varphi \in C^1$, $\varphi : \mathbb{R} \to \mathbb{R}$, and we require that $y(x_{tf}) = \varphi(x_{tf})$?

→ this means that the "terminal point b" is also free !

Exercise: Derive nec. conditions for weak extremum given (a, y_a) and $\varphi : \mathbb{R} \to \mathbb{R}$.

Hint: besides E.L. eq., you must obtain another condition accounting for variations in "x_{tf}" which explicitly involves φ'.

- we call it "transversality condition".