Lecture 4

Tuesday, April 7, 2026 5:28 PM

Review:

- Variational problem:

feasible subset

$$\min J(y) \quad \text{s.t.} \quad y \in A \subset V \rightsquigarrow \text{infinite-dim space.}$$

functional on V

- $V = \left(C^0([a,b], \mathbb{R}^n), \|\cdot\|_0\right)$ or $\left(C^1([a,b], \mathbb{R}^n), \|\cdot\|_1\right)$

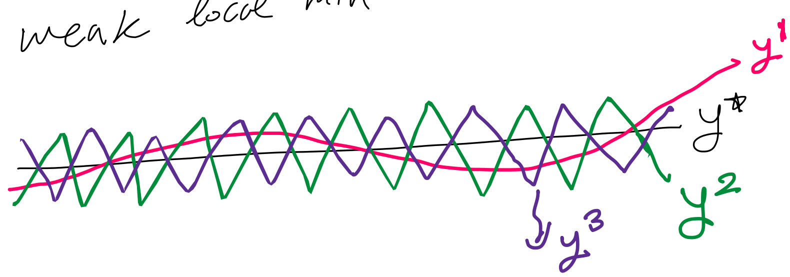
where $\|y\|_0 = \max_{t\in[a,b]} |y(t)|$ and $\|y\|_1 = \max_{t\in[a,b]} |y(t)| + \max_{t\in[a,b]} |y'(t)|$

- Strong vs **weak** local min y^*:

$\exists \varepsilon > 0 : \; J(y^*) \le J(y), \; \forall y : \|y - y^*\|_0$ vs 1 $< \varepsilon$

$\begin{cases} \|\cdot\|_0 \rightsquigarrow \text{strong local min} \\ \|\cdot\|_1 \rightsquigarrow \text{weak local min} \end{cases}$

Example:



$\rightsquigarrow$ y^1, y^2, y^3 are close to y^* in $\|\cdot\|_0$.

$\rightsquigarrow$ y^* is close to y^1 in $\|\cdot\|_1$ but <u>not</u> to y^2 or y^3.

$\rightsquigarrow$ y^2 is <u>not even</u> close to y^3 in $\|\cdot\|_1$.

- First and second variations (Gateaux derivatives)

$\rightsquigarrow$ $\delta J|_y(\eta) \triangleq \lim_{\alpha \to 0} \dfrac{J(y+\alpha\eta) - J(y)}{\alpha}$

where $\delta J|_y(\cdot)$ is a <u>linear mapping</u> for all admissible <u>directions</u> η : $y + \alpha\eta \in A$ for small enough α.

$$\rightsquigarrow \delta^2 J|_y(\eta) \triangleq \lim_{\alpha \to 0} \frac{J(y+\alpha\eta) - J(y) - \delta J|_y(\eta)\cdot\alpha}{\frac{1}{2}\alpha^2}$$

where $\delta^2 J|_y(\cdot)$ is a quadratic form for all feasible/admissible η.

- 1st-order and 2nd-order necessary cond.:

$\rightsquigarrow \delta J|_{y^*}(\eta) = 0$ for all η admissible.

$\rightsquigarrow \delta^2 J|_{y^*}(\eta) \geq 0$ " " " .

- 2nd-order sufficient cond:

  - we don't have it $\rightsquigarrow$ need to strengthen Gateaux deriv.

$\rightarrow$ Exercise 2.2 from the textbook.

Basic calc. of variation problem: (Formal)

- let $L: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the running cost / Lagrangian.
- Fix $a, b \in \mathbb{R}$, $y_a, y_b \in \mathbb{R}^n$.

position velocity

$$\min \int_a^b L(x, y(x), y'(x)) \cdot dx$$

s.t. $y \in C^1([a,b])$, $y(a) = y_a$, $y(b) = y_b$. ($\rightsquigarrow$ $y \in A \subset V$)

- $L = L(x, y, z)$ is a function of 3-variables, and should be <u>differentiable enough</u>! we denote $L_x, L_y, L_z, L_{xx}, \ldots$ the partial derivatives.

→ Admissible direction η? $y + \alpha\eta \in A$, $\forall \alpha$ small enough.

$\Rightarrow$ $\eta \in C^1([a,b], \mathbb{R}^n)$, $\eta(a) = \eta(b) = 0$. ✓

→ First variation $\delta J|_y(\eta)$?

Recall: $J(y + \alpha\eta) = J(y) + \delta J|_y(\eta) \cdot \alpha + o(\alpha)$.

$$J(y+\alpha\eta) = \int_a^b L(x, y+\alpha\eta, y' + \alpha\eta')\, dx$$

by Taylor expansion of L

$$= \int_a^b \Big[L(x, y, y') + L_y(x,y,y')\,\alpha\,\eta(x) + L_z(x, y, y')\,\alpha\,\eta'(x) + o(\alpha) \Big]\, dx$$

$$= J(y) + \alpha \int_a^b \big[L_y(x,y,y')\,\eta + L_z(x,y,y') \cdot \eta' \big]\, dx + o(\alpha)$$

$$\Rightarrow \delta J|_y(\eta) = \int_a^b \big[L_y(x,y,y')\,\eta + L_z(x,y,y')\,\eta' \big]\, dx$$

→ Is this linear in η?! Yes! why?

by integration by part:

$$\delta J|_y(\eta) = \int_a^b L_y(x,y,y')\eta\, dx - \int_a^b \left[\frac{d}{dx} L_z(x,y,y')\right]\eta\, dx$$

$$+ L_z(x,y,y')\cdot \eta \Big]_a^b$$

$$(\text{why?}) \quad = \int_a^b \left[L_y(x,y,y') - \frac{d}{dx} L_z(x,y,y')\right]\eta\, dx.$$

$\rightsquigarrow$ 1st-order nec. cond: $\delta J|_y(\eta) = 0 \quad \forall\, \eta$ admissible!

Lemma: (1st-fundamental lemma of cal. of variation)

If $\int_a^b \ell(x)\cdot\eta(x)\, dx = 0 \quad \forall\, \eta \in C^1$ such that

$\eta(a) = \eta(b) = 0$, and $\ell : [a,b] \to \mathbb{R}$ is continuous

Then $\ell(x) \equiv 0$ on $[a,b]$.

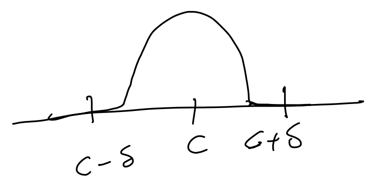
Proof: suppose not! w.l.o.g. suppose $\ell(c) > 0$ for some $c \in [a,b]$. but $\ell(x)$ is continuous, so

$\exists\, \varepsilon, \delta > 0 \quad : \ell(x) > \varepsilon, \ \forall\, x : |x-c| < \delta$.

Now, choose $\eta(x) \in C^1[a,b]$ such that

$$\begin{cases} \eta(c) = 1, \quad \eta(c-\delta) = \eta(c+\delta) = 0 \\ \eta(x) > 0, \quad \forall x : |x-c| < \delta \\ \eta(x) \equiv 0, \quad \forall x : |x-c| > \delta \\ \int_a^b \eta(x)\, dx = 1 \end{cases}$$

$\rightsquigarrow$ called "bump function"



$$\Rightarrow \int_a^b \ell(x)\,\eta(x)\,dx = \int_{c-\delta}^{c+\delta} \ell(x)\,\eta(x)\,dx$$

$$\geq \varepsilon \int_{c-\delta}^{c+\delta} \eta(x)\,dx = \varepsilon > 0 \quad \times$$

$\blacksquare$

So, $\ell(x) \equiv 0$ on $[a,b]$.

Back to 1st-order nec. cond.:

the Lemma implies:

$$\boxed{\mathcal{L}_y(x, y(x), y'(x)) - \frac{d}{dx}\mathcal{L}_z(x, y(x), y'(x)) \equiv 0, \quad \forall x \in [a,b]}$$

"the Euler-Lagrange Equation"

$\rightarrow$ So, if $y^*(x)$ satisfies E.L. equ. we call it an <u>extremal</u> <u>of functional J</u>.

→ E.L. equation is a 2nd-order differential equation with boundary cond. $y(a) = y_a$, $y(b) = y_b$. So, if a solution exists to EL eq. exists and is unique it should be the solution to our problem $\min_{y \in A} J(y)$.

But, existence of a solution is not trivial.

Example: $J(y) = \int_0^1 y(x)(y'(x))^2 dx$, $y(0) = y(1) = 0$.

then E.L. eq: $(y'(x))^2 - \frac{d}{dx}\left(2y(x) \cdot y'(x)\right) = 0$

Now, show that $y(x) \equiv 0$ is the only solution to E.L. eq. But, $y \equiv 0$ is neither a min nor a max of $J(y)$.

Aspects of EL Eq: Smoothness and special cases

- Smoothness requirement

$$L_y = \frac{d}{dx} L_z = L_{zx} + L_{zy} \cdot y' + L_{zz} \cdot y''$$

so it seems like we need to assume that both L and y are C^2, rather than C^1.

But, if we update our argument we see that C^1 smoothness assumption suffices and in fact we recover C^2 smoothness as a result; i.e., the extremal curves y will automatically enjoy more C^2-smoothness!

Lets update our argument: Recall

$$\delta J|_y(\eta) = \int_a^b \left[L_y(x,y,y')\eta + L_z(x,y,y')\eta' \right] dx$$

(Integration by part for $L_y \cdot \eta$)

$$= -\int_a^b \left(\int_a^x L_y(s,y(s),y'(s))\, ds \right) \cdot \eta'(x)\, dx + \left(\int_a^x L_y\, ds \right) \eta(x) \Big]_a^b + \int_a^b L_z \cdot \eta'(x)\, dx$$

(why?)

$$= \int_a^b \left[L_z(x,y(x),y'(x)) - \int_a^x L_y(s,y(s),y'(s))\, ds \right] \eta'(x)\, dx$$

1.st-order nec. cond: $\delta J|_y(\eta) = 0 \quad \forall \eta$ admissible.

Lemma 2: (2nd Fundamental Lemma of Calc. of variations)

Supp. $\int_a^b \xi(x)\eta'(x)\, dx = 0 \quad \forall \eta \in C^1([a,b])$, $\eta(a) = \eta(b) = 0$

If $\xi : [a,b] \to \mathbb{R}$ is continuous, then ξ is a constant func.

Proof: Exercise!

Applying this Lemma implies that:

Constant!

$$\mathcal{L}_z(x, y(x), y'(x)) = \int_a^x \mathcal{L}_y(s, y(s), y'(s))\, ds + C$$

$\Rightarrow$ if $\mathcal{L}_y$ is continuous then $\mathcal{L}_z$ is C^1 and $\frac{d}{dx}\mathcal{L}_z = \mathcal{L}_y$.

So we don't need any assumption more that $\eta, \mathcal{L} \in C^1$.

Generalization: our analysis can be similarly generalized to show that the E.L. eq. even applies under piece-wise C^1 assumption. (continuous curves on $[a,b]$ w/ continuous derivative everywhere except finitely many points).

Exercise: How? Generalize Lemma 2!

- special cases of E.L. eq. for $\mathcal{L}(x,y,y')$:

$$\mathcal{L}_y = \frac{d}{dx}\mathcal{L}_z = \mathcal{L}_{zx} + \mathcal{L}_{zy}\cdot y' + \mathcal{L}_{zz}\cdot y''.$$

Case 1: $\mathcal{L}$ has "no y" direct dependence

$$\mathcal{L} = \mathcal{L}(x, y') \Rightarrow \mathcal{L}_y \equiv 0 \Rightarrow \frac{d}{dx}\mathcal{L}_z \equiv 0.$$

1st order differential equation $\Rightarrow$ $$\boxed{L_z(x, y'(x)) \equiv C \quad \text{for any constant } C \in \mathbb{R}.}$$

" $L_z(x, y'(x))$ " is called the "<u>momentum</u>"

<u>Case 2:</u> L has "<u>no x</u>" direct dependence

$$L = L(y, y') \Rightarrow L_y = L_{zy}\cdot y' + L_{zz}\cdot y''$$

(multiply by y') $\Rightarrow$ $$L_y \cdot y' = L_{zy}(y')^2 + L_{zz}\cdot y''\cdot y'$$

$$\Rightarrow L_{zy}(y')^2 + L_{zz}\cdot y''\cdot y' - L_y\cdot y' = 0$$

Compute $\frac{d}{dx}(L_z\cdot y' - L) =$

$$\cancel{L_{zx}}\cdot y' + L_{zy}\cdot y'\cdot y' + L_{zz}\cdot y''\cdot y' + \underline{L_z y''} - \cancel{L_x} - L_y y' - \underline{L_z\cdot y''}$$

$$= L_{zy}\cdot(y')^2 + L_{zz}\, y''\cdot y' - L_y\cdot y' = 0$$

$$\Rightarrow \boxed{\begin{gathered} L_z(x, y(x), y'(x))\cdot y'(x) - L(x, y(x), y'(x)) \equiv C \\ \text{for any constant } C \in \mathbb{R} \end{gathered}}$$

" $L_z\cdot y' - L$ " is called the "<u>Hamiltonian</u>"

Reading Assignment: Chapter 2.1, 2.2, 2.3