

Example: minimum time optimal control and the switching curves

$$\ddot{x} = u, \quad u \in [-1, 1] \quad \text{'controlling acceleration'}$$

$$x_0 \neq 0 \text{ given}, \quad x_f = 0 \text{ fixed}$$

Problem: drive $x(t)$ to $x_f = 0$ in minimum time.

$$\min_{u(t) \in [-1, 1]} J(u) = \int_0^{t_f} 1 \, dt = t_f \Rightarrow \underline{L \equiv 1}.$$

subject to the dynamics: $\begin{matrix} x_1 \leftarrow \text{position} \\ x_2 \leftarrow \text{velocity} \end{matrix} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}.$

Pontryagin's maximum principle:

$$H = \langle p, \dot{x} \rangle - p_0 L = p_1 x_2 + p_2 u - p_0.$$

$$\underline{\dot{p} = -H_x|_q} \Rightarrow \begin{pmatrix} \dot{p}_1^* \\ \dot{p}_2^* \end{pmatrix} = \begin{pmatrix} -H_{x_1}|_q \\ -H_{x_2}|_q \end{pmatrix} = \begin{pmatrix} 0 \\ -p_1^* \end{pmatrix}.$$

$$p_1^* \equiv \text{constant.} \quad \text{and} \quad p_2^* = \underbrace{-p_1^* t + p_2^*(0)}_{\text{constants}} \leftarrow \text{linear function of time.}$$

$$\underline{\dot{x} = H_p|_q} \Rightarrow \begin{pmatrix} \dot{x}_1^* \\ \dot{x}_2^* \end{pmatrix} = \begin{pmatrix} x_2^* \\ u \end{pmatrix}$$

$u \mapsto H(x^*(t), u, p^*(t))$ is maximized at $u = u^*(t) \Rightarrow$

$$u^*(t) = \begin{cases} 1 & \text{if } p_2^*(t) > 0 \\ -1 & \text{if } p_2^*(t) < 0 \\ ? & \text{if } p_2^*(t) = 0 \end{cases}$$

claim: $P_2^*(t)$ cannot vanish on an interval for time-optimal control prob.

because $H|_{\dot{x}} = H_{\dot{x}}|_{\dot{x}} = 0$, bc system is LTI.

if $P_2^*(t) = 0$ on $I \Rightarrow \begin{pmatrix} P_1^* \\ P_2^* \end{pmatrix} = 0$ on $I \Rightarrow H|_{\dot{x}} = -P_0 = 0 \Rightarrow P_0 = 0$

but this contradicts maximum principle of $(P, P_0) \neq (0, 0)!$ $\dot{x}$.

claim: $P_2^*(t)$ has at most one zero? why? because it is linear in t .

Thus, $u^*(t)$ is well-defined except potentially at a single t .

$\Rightarrow$ optimal $u^* = \pm 1$ with at most one switch $\leadsto$ "bang-bang control"

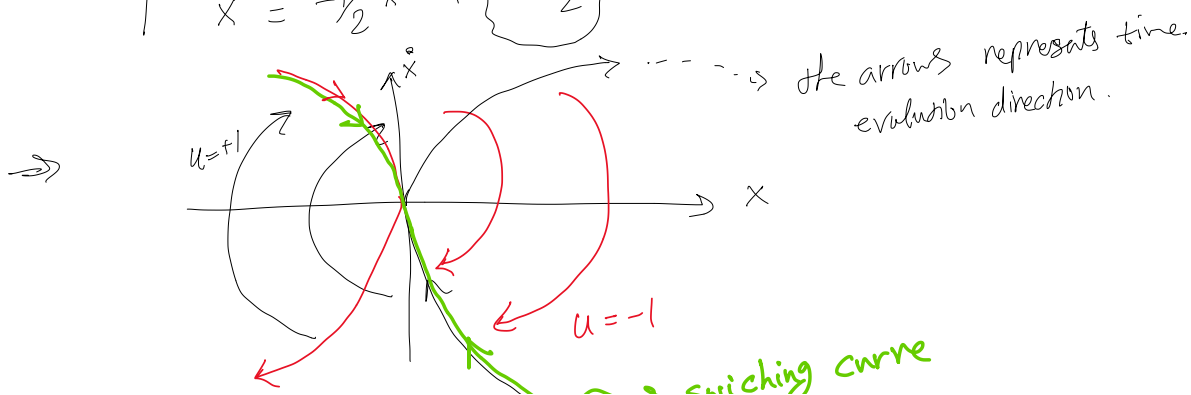
the optimal trajectory:

$$\ddot{x} = u \Rightarrow \dot{x}_{(t)} = ut + a \Rightarrow x_{(t)} = \frac{1}{2} ut^2 + at + b$$

$$\Rightarrow \begin{cases} x(t) = \frac{1}{2} t^2 + at + b, & \dot{x}_{(t)} = t + a & \text{if } u = +1 \\ x(t) = -\frac{1}{2} t^2 + at + b, & \dot{x}_{(t)} = -t + a & \text{if } u = -1 \end{cases}$$

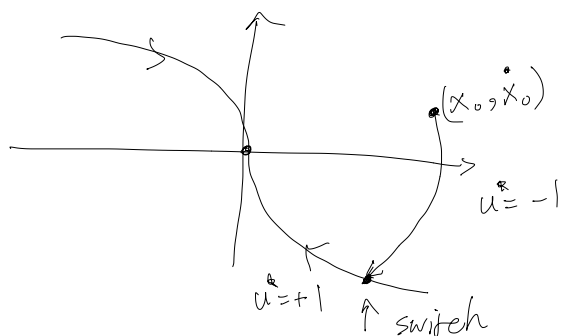
So, in $(x, \dot{x})$ coordinates:

$$\Rightarrow \begin{cases} x = \frac{1}{2} \dot{x}^2 + \left(b - \frac{a^2}{2}\right) & \text{if } u = +1 \\ x = -\frac{1}{2} \dot{x}^2 + \left(b - \frac{a^2}{2}\right) & \text{if } u = -1 \end{cases}, \text{ for some } C \in \mathbb{R}.$$

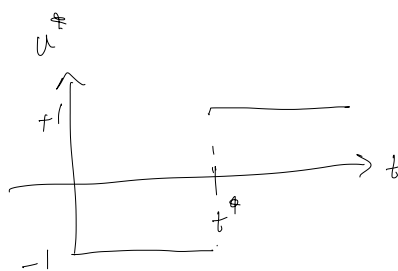




Example:
⇒



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Exercise: Read about the general LTI Bang-Bang control in sections 4.4.2.

Then nonlinear case is treated in sec. 4.4.3.

Dynamic Programming and Hamilton-Jacobi-Bellman eq:

→ An alternative theory to optimal control using Principle of optimality

Consider the fixed-time free-end point optimal control problem

$$\min_{u(\cdot)} J(t_0, x_0, u) = \int_{t_0}^{t_f} L(s, x(s), u(s)) ds + k(x(t_f))$$

$$\text{s.t. } \dot{x} = f(s, x, u), \quad x(t_0) = x_0.$$

Dynamic programming instead solves a family of minimizations

for each $t \in [t_0, t_f]$:

$$\begin{cases} \min_{u(\cdot)} J(t, x_t, u) = \int_t^{t_f} L(s, x(s), u(s)) ds + k(x(t_f)) \\ \text{s.t. } \dot{x} = f(s, x, u), \quad x(t) = x_t \end{cases}$$

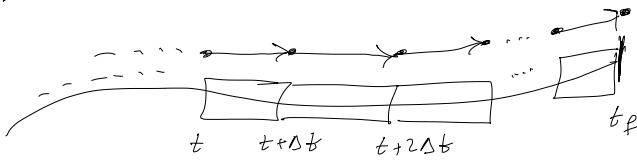
$\downarrow$ initial time
 $\downarrow$ initial state

⇒ value function: $V(t, x_t) \triangleq \inf_{u[t, t_f]} J(t, x_t, u)$.

Then obviously, $V(t_f, x) = k(x)$, $\forall x \in \mathbb{R}^n$ (i.e. for all possible x_f 's)

more generally, $V(t, x) = K(x)$, $\forall (t, x) \in S \subset [t_0, \infty) \times \mathbb{R}^n \rightarrow$ Boundary cond.

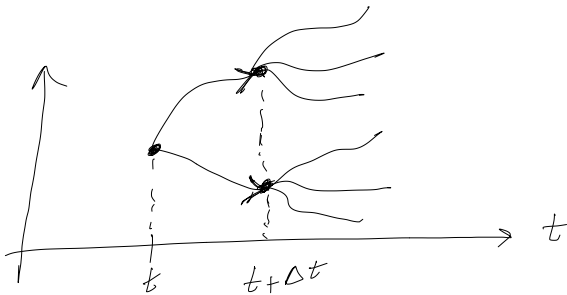
→ The Directed Acyclic Graph (DAG) for subproblems in optimal control:



→ Principle of optimality: $\forall (t, x) \in [t_0, t_f] \times \mathbb{R}^n$, $\forall \Delta t \in (0, t_f - t]$

$$V(t, x) = \inf_{u[t, t+\Delta t]} \left\{ \int_t^{t+\Delta t} L(s, x(s), u(s)) ds + V(t+\Delta t, x(t+\Delta t)) \right\} \triangleq \bar{V}(t, x)$$

↑
trajectory starting from $x(t) = x$.



minimize this first on $[t+\Delta t, t_f]$ to get $V(t+\Delta t, x(t+\Delta t))$

then minimize this portion $[t, t+\Delta t]$ given $V(t+\Delta t, x(t+\Delta t))$

Note: for second minimization $x(t+\Delta t)$ is unknown and coupled the terms

$$\int_t^{t+\Delta t} L ds \text{ and } V(t+\Delta t, x(t+\Delta t)).$$

∴ $V(t, x) = \bar{V}(t, x)$, $\forall (t, x)$.

Q: Does Principle of optimality hold?

$$\begin{aligned} \bar{V}(t, x) &= \inf_{u[t, t+\Delta t]} \left\{ \int_t^{t+\Delta t} L ds + \inf_{u[t+\Delta t, t_f]} \int_{t+\Delta t}^{t_f} L ds \right\} \\ &= \inf_{u[t, t+\Delta t]} \left\{ \inf_{u[t+\Delta t, t_f]} \int_t^{t_f} L ds \right\} \\ &\geq \inf_{u[t, t_f]} \left\{ \int_t^{t_f} L ds \right\} = \underline{V}(t, x), \quad \forall (t, x). \end{aligned}$$

Conversely, $\forall \varepsilon > 0$, (by def. of $V(t, x)$) $\exists u_\varepsilon(\cdot)$ on $[t, t_f]$ such that.

$$V(t, x) + \varepsilon \geq J(t, x, u_\varepsilon), \quad \forall (t, x)$$

Then, $\forall (t, x)$:

$$J(t, x, u_\varepsilon) = \int_t^{t+\Delta t} L(s, x_\varepsilon(s), u_\varepsilon(s)) ds + J(t+\Delta t, x_\varepsilon(t+\Delta t), u_\varepsilon)$$

$$\inf_{u[t+\Delta t, t_f]} \geq \int_t^{t+\Delta t} L(s, x_\varepsilon(s), u_\varepsilon(s)) ds + V(t+\Delta t, x_\varepsilon(t+\Delta t))$$

$$\inf_{u[t, t+\Delta t]} \geq \bar{V}(t, x).$$

$$\Rightarrow \forall \varepsilon > 0: V(t, x) + \varepsilon \geq J(t, x, u_\varepsilon) \geq \bar{V}(t, x), \quad \forall (t, x)$$

$$\Rightarrow \underline{V}(t, x) \geq \bar{V}(t, x), \quad \forall (t, x).$$

Hamilton-Jacobi-Bellman (HJB) Equation:

Using Principle of optimality to solve for $V(t, x)$:

$$x(t+\Delta t) = x + f(t, x, u(t)) \cdot \Delta t + o(\Delta t), \quad \text{recall } x(t) = x.$$

$$x(t+\Delta t) = x + f(t, x, u(t)) \cdot \Delta t + o(\Delta t)$$

$$\Rightarrow V(t+\Delta t, x(t+\Delta t)) = V(t, x) + V_t(t, x) \cdot \Delta t + \langle V_x(t, x), f(t, x, u(t)) \rangle \Delta t + o(\Delta t)$$

(assuming that V is C^1).

$$\Rightarrow \int_t^{t+\Delta t} L(s, x(s), u(s)) ds = L(t, x, u(t)) \cdot \Delta t + o(\Delta t)$$

Princ. of opt.

$$\Rightarrow V(t, x) = \inf_{u[t, t+\Delta t]} \left\{ L(t, x, u(t)) \cdot \Delta t + \cancel{V(t, x)} + V_t(t, x) \cdot \Delta t + \langle V_x(t, x), f(t, x, u(t)) \rangle \cdot \Delta t + o(\Delta t) \right\}$$

$$\Rightarrow 0 = \inf_{u[t, t+\Delta t]} \left\{ L(t, x, u(t)) \Delta t + V_t(t, x) \cdot \Delta t + \langle V_x(t, x), f(t, x, u(t)) \rangle \Delta t + o(\Delta t) \right\}$$

now, divide by Δt :

only depends on $u(t)$

$$0 = \inf_{u[t, t+\Delta t]} \left\{ L(t, x, u(t)) + V_t(t, x) + \langle V_x(t, x), f(t, x, u(t)) \rangle + \frac{o(\Delta t)}{\Delta t} \right\}$$

$u(s), s > t$
 $\downarrow$

now, take $\Delta t \rightarrow 0$:

"HJB PDE"

$$-V_t(t, x) = \inf_{u \in U} \left\{ L(t, x, u) + \langle V_x(t, x), f(t, x, u) \rangle \right\}$$

on $[t_0, t_f) \times \mathbb{R}^n$, with Boundary Cond. $V(t, x) = K(x)$, $\forall (t, x) \in S$

Rewriting HJB using Hamiltonian:

Recall $H(t, x, u, p) = \langle p, f(t, x, u) \rangle - L(t, x, u)$

recall $V(t, x) = \inf_{u \in U} \{ \dots \}$

$$\Rightarrow V_t(t, x) = \sup_{u \in U} \left\{ -L(t, x, u) + \langle -V_x(t, x), f(t, x, u) \rangle \right\}$$

$$\Rightarrow \boxed{V_t(t, x) = \sup_{u \in U} H(t, x, u, -V_x(t, x))}$$

So far, we did not assume existence of the optimal control $\underline{u}^*$, but if it exists, then it generates $\underline{x}^*$ and we have the value function along the optimal trajectory:

$$\begin{aligned} -V(t, x^*(t)) &= \min_{u \in U} \left\{ L(t, x^*(t), u) + \langle V_x(t, x^*(t)), f(t, x^*(t), u) \rangle \right\} \\ &= L(t, x^*(t), u^*(t)) + \langle V_x(t, x^*(t)), f(t, x^*(t), u^*(t)) \rangle \\ &= -H(t, x^*(t), u^*(t), -V_x(t, x^*(t))) \end{aligned}$$

Now, HJB along the optimal trajectory reads:

$$H(t, x^*(t), u^*(t), -V_x(t, x^*(t))) = \max_{u \in U} H(t, x^*(t), u, -V_x(t, x^*(t)))$$

$\hookrightarrow$ this should resemble the maximum principle!

Sufficient condition for optimality:

suppose (1) $\exists$ C^1 -function: $\hat{V}_{(x)}: [t_0, t_f] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying HJB:

$$\hat{V}_t(t, x) = \sup_{u \in U} H(t, x, u, -\hat{V}_x(t, x)), \quad \forall (t, x) \in [t_0, t_f) \times \mathbb{R}^n$$

and $\hat{V}(t_f, x) = K(x) \quad \forall (t_f, x) \in \mathcal{S}_f$.

suppose (2) $\exists \hat{u}(t): [t_0, t_f] \rightarrow U$ corresponding to trajectory $\hat{x}(t)$

with $\hat{x}(t_0) = x_0$ and s.t. $\forall t$:

$$H(t, \hat{x}(t), \hat{u}(t), -\hat{V}_x(t, \hat{x}(t))) = \max_{u \in U} H(t, \hat{x}(t), u, -\hat{V}_x(t, \hat{x}(t))).$$

$\Rightarrow$ Then, $\hat{V}(t_0, x_0)$ is the optimal cost, $\hat{V}$ is the optimal value function and $\hat{u}$ is the globally optimal control. (not claimed to be unique!).

Proof: $\hat{u}(t)$ attains the inf/sup in HJB for $\hat{x}(t)$. Thus

$$\begin{aligned} \hat{V}_t(t, \hat{x}(t)) &= H(t, \hat{x}(t), \hat{u}(t), -\hat{V}_x(t, \hat{x}(t))) \\ &= - \underbrace{\langle \hat{V}_x(t, \hat{x}(t)), f(t, \hat{x}(t), \hat{u}(t)) \rangle}_{=0} - L(t, \hat{x}(t), \hat{u}(t)) \end{aligned}$$

$$\Rightarrow \frac{d}{dt} (\hat{V}(t, \hat{x}(t))) + L(t, \hat{x}(t), \hat{u}(t)) = 0$$

$$\Rightarrow \hat{V}(t_f, \hat{x}(t_f)) - \hat{V}(t_0, \hat{x}_0(t_0)) + \int_{t_0}^{t_f} L(t, \hat{x}(t), \hat{u}(t)) dt = 0$$

$$\Rightarrow V(t_0, x_0) = \int_{t_0}^{t_f} L(t, \hat{x}(t), \hat{u}(t)) dt + K(\hat{x}(t_f)) = J(t_0, x_0, \hat{u}) \quad \textcircled{I}$$

on the other hand if $x(t)$ is any other trajectory for any other control $u(t)$, we can similarly obtain

$$\hat{V}(t_0, x_0) \leq \int_{t_0}^{t_f} L(t, x(t), u(t)) dt + K(x(t_f)) = J(t_0, x_0, u). \quad \textcircled{II}$$

Comparing $\textcircled{I}$ and $\textcircled{II}$ proves that $\hat{u}(\cdot)$ is a globally optimal control and

$\hat{V}(t_0, x_0)$ is the optimal cost. $\equiv$

note: with the same argument we can show that $\hat{V}(t, \hat{x}(t))$ is the optimal cost-to-go from any point $(t, \hat{x}(t))$ on the optimal trajectory $\hat{x}$. But, (t_0, x_0) could be any initial condition and $\hat{V}(t_0, x_0)$ can be defined for all such (t_0, x_0) .

$\Rightarrow$ this gives a family of optimal controllers $\hat{u}$ where each $\hat{u}(t_0)$ is optimal for specific (t_0, x_0) , resulting in feedback optimal control.

Exercise : For LQR problem, find the family of $\hat{V}$ and the optimal feedback controllers.

Recall: $J(u) = \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt$ s.t. $\dot{x} = Ax + Bu, x(t_0) = x_0$

$Q \in \mathbb{R}^{n \times n}$ positive semidefinite s.t.

show that $\exists P(t): [t_0, t_f] \rightarrow \mathbb{R}^{n \times n}$

$\hat{V}(t, x) = x^T P(t) x$ is the value function. and

$\hat{u}(t) = -K(t) \hat{x}(t)$ for some $K(t): [t_0, t_f] \rightarrow \mathbb{R}^{m \times n}$.

Find an ODE for $P(t)$ and a formula for $K(t)$.

(notice that $\hat{u}(t)$ is a feedback controller)

→ see chapter 6.

→ Under what conditions does $P(t), K(t)$ exists as $t_f \rightarrow \infty$?

} see Theorem 6.1:

If (A, B) controllable, $(A, Q^{1/2})$ observable, and $R > 0$ P.D..

Then ...