Lecture 15

Tuesday, May 19, 2026 3:02 PM

Review: Free-time fixed-endpoint optimal control

$$\min_{u(\cdot)} J(u) = \int_{t_0}^{t_f} L(t,x,u)\,dt + K(t_f, x_{t_f})$$

s.t. $\dot{x} = f(t,x,u)$, $x_{t_0} = x_0$

$u(t) \in \mathbb{R}^m$, $\forall t$.

$(t_f, x_{t_f}) \in S = [t_0, \infty) \times \{x_f\}$

→ Pontryagin's maximum principle: $H = \langle p, f \rangle - L$

① $\begin{cases} \dot{x}^* = H_p|_* & x(t_0) = x_0, \quad x(t_f) = x_f \\ \dot{p}^* = -H_x{}^* & p(t_f) \text{ is free.} \end{cases}$

② $u \mapsto H(x^*(t), u, p^*(t))$ is maximized at $u = u^*(t)$, $\forall t$.

③ $H|_*(t) \equiv 0$, $\forall t$.

Basic Free-time variable-endpoint optimal control:

→ $S = [t_0, \infty) \times S_1$ where S_1 is a $\underset{k \le n}{\underline{k\text{-dimensional}}}$ surface $\subset \mathbb{R}^n$.

$$S_1 \triangleq \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots = h_{n-k}(x) = 0\}, \quad h_i \in C^1.$$

→ Problem is time-invariant and terminal cost $K \equiv 0$.

* Pontryagin's maximum Principle:

If $u^*: [t_0, t_f] \to U$ is optimal control with $x^*: [t_0, t_f] \to \mathbb{R}^n$ optimal traj.

Then, $\exists\ P^*_{(\cdot)}: [t_0, t_f] \to \mathbb{R}^n$ and $\exists\ P_0^* \leq 0$ Constant s.t. $(P_0^*, P(t)) \neq (0,0)\ \forall t$

and for $H = H(x, u, P, P_0) = \langle P, f(x,u) \rangle + P_0 \cdot L(x,u)$.

① $\dot{x}^* = H_p \big|_*$, $\dot{p}^* = -H_x \big|_*$, $x^*(t_0) = x_0$, $x^*(t_f) \in S_1$.

② $H(x^*(t), u^*(t), p^*(t), P_0^*) \geq H(x^*(t), u, p^*(t), P_0^*)$, $\forall t$, $\forall u$.

③ $H(x^*(t), u^*(t), p^*(t), P_0^*) = 0$, $\forall t$.

④ vector $p^*(t_f)$ is orthogonal to the tangent space to S_1 at $x^*(t_f)$

$$\langle p^*(t_f), d \rangle = 0 \quad , \quad \forall d \in T_{x^*(t_f)} S_1 .$$

$\to$ called "transversality condition" which should resemble the condition on the variational problem with $y(x_f) = \varphi(x_f)$ for a given φ.

$$T_{x^*(t_f)} S_1 = \{ d \in \mathbb{R}^n : \langle \nabla h_i (x^*(t_f)), d \rangle = 0 \ , \ \forall i = 1, \dots, n-k \}.$$

$\to$ If $S_1 = \mathbb{R}^n \Rightarrow T_{x^*(t_f)} S_1 = \mathbb{R}^n \Rightarrow p^*(t_f) = 0$ because we assumed $K \equiv 0$

$\to$ If $S_1 = \{x_f\} \Rightarrow T_{x^*(t_f)} S_1 = 0 \Rightarrow p^*(t_f)$ is free!

Note: The constant P_0^* can be set to $\underline{-1}$ if the u^* is a non-degenerate / regular optimal control and otherwise it is set to $\underline{0}$.

→ what if $x(t_f) \in S_1$ and we still have some final state cost $K \neq 0$?

Recall the trick of embedding "K" term into the Lagrangian L.

→ what if the system is time-varying?

Recall the trick of embedding time as a new state "$x_{n+1} = t$".

- See Lectures from the author for the proof: 10/22/20 and 10/27/20.

- Next week (I'm traveling): watch Lectures 10/29/20 and 11/05/20

→ see the Table of Pontryagin's maximum/minimum principle!