Lecture 14

Friday, May 8, 2026 8:02 AM

Review: Fixed-time Free-endpoint Problem

$$\min_{u(\cdot)} \quad J(u) = \int_{t_0}^{t_f} L(t,x,u)\,dt + K(t_f, x_{t_f})$$

s.t. $\quad \dot{x} = f(t,x,u), \quad x_{t_0} = x_0$

$\quad u(t) \in \mathbb{R}^m, \ \forall t.$

$\quad (t_f, x_{t_f}) \in S = \{t_f\} \times \mathbb{R}^n$

$$\boxed{H(t,x,u,p) \triangleq \langle p, f(t,x,u) \rangle - L(t,x,u)}$$

$$J(u) = \int_{t_0}^{t_1} \Big[\langle p(t), \dot{x}(t) \rangle - H(t, x(t), u(t), p(t)) \Big] dt + K(x(t_1))$$

perturbation: $u = u^* + \alpha \xi \Rightarrow x = x^* + \alpha \eta + o(\alpha)$ and

$$\delta J|_{u^*}(\xi) = \lim_{\alpha \to 0} \frac{J(u^* + \alpha\xi) - J(u^*)}{\alpha} \quad \Rightarrow$$

$$\delta J|_{u^*}(\xi) = -\int_{t_0}^{t_1} \Big[\langle \dot{p} + H_x(t,x^*,u^*,p), \eta \rangle + \langle H_u(t,x^*,u^*,p), \xi \rangle \Big] dt$$

$$+ \langle K_x(x^*(t_1)) + p(t_1), \eta(t_1) \rangle$$

where $\quad \dot{\eta}_t = f_x|_* \cdot \eta_t + f_u|_* \cdot \xi_t, \quad \eta(t_0) = 0.$

- $\left(\delta J|_{u^*}(\xi) = 0, \ \forall \xi \text{ admissible}\right)$

 1st-order nec. conditions for Fixed-time Free-endpoint?

$$\boxed{\begin{aligned} &\dot{x}^* = H_p\big|_*, \quad x^*(t_0) = x_0 \\ &\dot{p}^* = -H_x\big|_*, \quad p^*(t_1) = -K_x(x^*(t_1)) \\ &H_u\big|_* = 0, \quad \forall t \in [t_0, t_1] \end{aligned}}$$

- $\left(\delta^2 J\big|_{u^*}(\xi) \geq 0, \ \forall \xi \text{ admissible}\right)$

2nd-order nec. conditions for Fixed-time Free-endpoint:

$$\delta^2 J\big|_{u^*}[\xi] = -\frac{1}{2}\int_{t_0}^{t_1} \begin{pmatrix} \eta \\ \xi \end{pmatrix}^T \cdot \begin{pmatrix} H_{xx} & H_{xu} \\ H_{xu} & H_{uu} \end{pmatrix}\Bigg|_* \cdot \begin{pmatrix} \eta \\ \xi \end{pmatrix} dt + \frac{1}{2}\eta^T \cdot K_{xx}(x^*(t_1)) \cdot \eta$$

where $\dot{\eta} = (f_x\big|_*)\eta + (f_u\big|_*)\xi, \quad \eta(t_0) = 0$.

$\Rightarrow \boxed{H_{uu}\big|_* \preceq 0 \quad \text{N.S.D. for all } t.}$

Note: This is only necessary and NOT sufficient for

$\delta^2 J\big|_{u^*}(\xi) > 0, \ \forall \xi \not\equiv 0$.

Note: $\frac{d}{dt}(H)\big|_* = H_t\big|_* \quad \forall t.$

In particular, for time-invariant problems $H \equiv \text{Constant}$!

Basic <u>Time-invariant</u> <u>Free-time</u> <u>Fixed-endpoint</u> :

→ $S = [t_0, \infty) \times \{x_f\}$ for a fixed x_f.

→ Assume the problem is <u>time-invariant</u> → (Basic).

then $K = K(x(t_f)) = K(x_f) \leftarrow$ global constant

So, we just drop it from our analysis.

(*) <u>Maximum Principle</u> for this problem:

If $u^*: [t_0, t_f] \to U$ is a <u>non-degenerate</u> global optimal control,

corresponding to $x^*: [t_0, t_f] \to \mathbb{R}^n$ (optimal trajectory).

Then,

① $\exists\, p^*: [t_0, t_f] \to \mathbb{R}^n$ s.t. $\begin{cases} \dot{x}^* = H_p|_* \\ \dot{p}^* = -H_x|_* \end{cases}$

with BCs : $x(t_0) = x_0$ and $x(t_f) = x_1$, where

$$H = H(x, u, p) = \langle p, f(x,u) \rangle - L(x,u).$$

② For each fixed t, the function $u \mapsto H(x^*(t), u, p^*(t))$

has a global maximum at $u = u^*(t)$, i.e.,

$$H(x^*(t), u^*(t), p^*(t)) \geq H(x^*(t), u, p^*(t)), \quad \forall t \in [t_0, t_f], \ \forall u \in U.$$

③ $H(x^*(t), u^*(t), p^*(t)) \equiv 0$ for all $t \in [t_0, t_f]$.

(We knew that $H|_* =$ constant because $\frac{d}{dt} H|_* = H_t|_* = 0$

as long as $\dot{x}^* = H_p|_*$ and $\dot{p}^* = -H_x|_*$. But, this says

this constant is zero).

- If, u^* is a <u>degenerate</u> optimal control, then ①, ② and ③

holds for $H_{deg.} \triangleq \langle p, f(x,u) \rangle$, replacing <u>Hamiltonian H</u>!

(Notice that there is no "L" in the nec. conditions).

Let's try to argue this nec. condition is true:

$J(u) = \int_{t_0}^{t_f} L(x,u)\, dt \quad \rightsquigarrow$ no "K" term.

if $H = \langle p, f(x,u) \rangle - L \Rightarrow$

$$J(u) = \int_{t_0}^{t_f} (\langle p, \dot{x} \rangle - H)\, dt = \int_{t_0}^{t_f} -\langle \dot{p}, x \rangle - H)\, dt + \langle p, x \rangle \Big]_{t_0}^{t_f}$$

Suppose <u>$t_f = t_f^*$ fixed</u>. Consider perturbation $u = u^* + \alpha \xi$ then:

$$\Rightarrow \delta J|_{u^*}[\xi] = \int_{t_0}^{t_f} \left[-\langle \dot{p}, \eta(x) \rangle - \langle H_x(x^*, u^*, p), \eta(x) \rangle - \langle H_u(x^*, u^*, p), \xi \rangle \right] dt$$

$$+ \langle p(t_f), \cancel{\eta(t_f)} \rangle - \langle p(t_0), \cancel{\eta(t_0)} \rangle$$

$\qquad\qquad 0 \qquad\qquad\qquad 0$

because ξ is peicewise-C^1 $\qquad$ admissible η : $\eta(t_0) = \eta(t_f) = 0$.

and $x_{t_0} = x_0$ fixed

but $\lambda(t_f) = \cdots$ $\Big|$ $\Rightarrow$
$x(t_0) = x_0$

Now, choose p^* s.t. $\dot{p}^* = -H_x\big|_*$ and $\dot{x}^* = H_p\big|_*$. Then,

$$\delta J\big|_{u^*}(\xi) = -\int_{t_0}^{t_f} \langle H_u\big|_*, \xi\rangle\, dt = 0 \Rightarrow \underline{H_u\big|_* = 0 \quad \forall t \in [t_0, t_1].}$$

similarly, we can obtain that $\underline{H_{uu} \preceq 0 \text{ N.S.D } \forall t \in [t_0, t_1]}$.

(Close! but we still need to show $H(x^*, \cdot, p^*)$ is indeed maximized at u^*).

<u>How about ③?</u>

Note that t_f is free! So consider variations w.r.t. t_f as well:

but notice that $(\xi, \Delta t)$ is admissible only if $x(t_f + \alpha\Delta t) = x_f$, fixed.

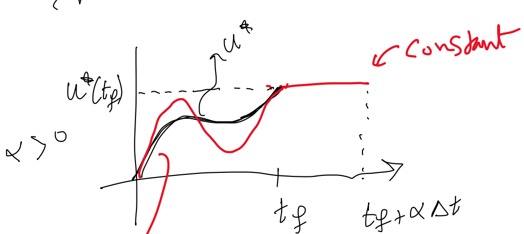
(x) <u>Idea:</u> $g(\alpha) = J(u^*, t_f + \alpha\Delta t) = \int_{t_0}^{t_f + \alpha\Delta t} L(x, u)\, dt \Rightarrow \frac{d}{d\alpha} g(\alpha)\Big|_{\alpha=0} = L(x_f^*, u_f^*) \cdot \Delta t$

$g'(0) = 0 \quad \forall \Delta t \Rightarrow L(x_f, u_f^*) = 0$ but this is <u>NOT True</u>! why?

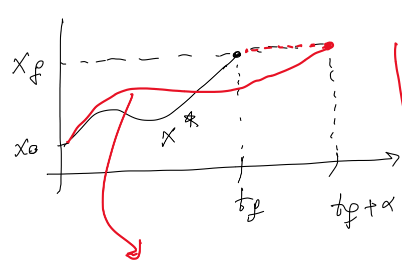
because we ignored the constraints $\dot{x} = f(x, u)$, $x(0) = x_0$, $x(t_f) = x_f$

Instead, we look at the "augmented Lagrangian" with <u>Hamiltonian</u>!

→ The other issue: $u^*: [t_0, t_f] \to U$, what is $u^*(t_f + \alpha\Delta t)$?!



$\Rightarrow$



$X(t_0) = X_0$,

$\boxed{X(t_f + \alpha\Delta t) = X_f}$?

$\Big\{ u^* + \alpha\xi$ on $[t_0, t_f]$

$\dot{x} = \Big\{ f(x, u^* + \alpha\xi)$ on $[t_0, t_f]$

$$u = \begin{cases} u^*(t_f) & \text{on } [t_f, t_f+\alpha\Delta t] \end{cases} \quad \Big| \; f(x, u^*(t_f)) \quad \text{on } [t_f, t_f+\alpha\Delta t]$$

$$g(\alpha) = J(u^* + \alpha\xi, t_f^* + \alpha\Delta t) = \int_{t_0}^{t_f+\alpha\Delta t} \Big[\langle p, \dot{x}\rangle - H(x^* + \alpha\eta + o(\alpha), u^* + \alpha\xi, p)\Big] dt$$

$$= \int_{t_0}^{t_f+\alpha\Delta t} \big(-\langle \dot{p}, x\rangle - H\big)\, dt + \langle p, x\rangle \Big|_{t_0}^{t_f+\alpha\Delta t}$$

$$\begin{cases} H_u|_* = 0 & \text{on } [t_0, t_f] \\ \alpha\xi = 0 & \text{on } t \geq t_f. \end{cases}$$

$$= \int_{t_0}^{t_f+\alpha\Delta t} \Big(-\langle \dot{p}, x^* + \cancel{\alpha\eta}\rangle - H|_* - \cancel{\langle H_x|_*, \alpha\eta\rangle} - \langle \cancel{H_u|_*}, \alpha\xi\rangle\Big) dt$$

$\triangleq -\dot{p}^*$; ok

$$+ \langle p, x\rangle \Big|_{t_0}^{t_f} + o(\alpha)$$

$$= \int_{t_0}^{t_f+\alpha\Delta t} \big(-\langle \dot{p}, x^*\rangle - H\big) dt + \langle p(t_f + \alpha\Delta t), \underbrace{x(t_f+\alpha\Delta t)}_{= x_f \rightsquigarrow \text{fixed}}\rangle$$

$(\star)$

and not a func. of α.

$$- \langle p(t_0), x(t_0)\rangle + o(\alpha).$$

$$\frac{d}{d\alpha} g(\alpha)\Big|_{\alpha=0} = \Big[\cancel{-\langle \dot{p}^*(t_f), x^*(t_f)\rangle} - H|_*(t_f)\Big]\Delta t + \cancel{\langle \dot{p}^*(t_f), x^*(t_f)\rangle}\Delta t$$

$$= -H|_*(t_f) \cdot \Delta t = 0 \quad \text{for all admissible } \Delta t.$$

$$\Rightarrow \quad \boxed{H(x^*(t_f), u^*(t_f), p^*(t_f)) = 0}$$

Also : $\dfrac{d}{dt} H|_* = H_t|_* + \langle H_x|_*, \dot{x}^*\rangle + \langle H_u|_*, \dot{u}^*\rangle + \langle H_p|_*, \dot{p}^*\rangle$

$$= H_t|_* - \langle \dot{p}^*, \dot{x}^*\rangle + 0 \quad \langle \dot{x}^*, \dot{p}^*\rangle$$

$$= H_t \big|_* \quad = 0 \quad \text{if the problem is time-invariant.}$$

$$\Rightarrow \quad H\big|_* (t) = \text{Constant} = H\big|_* (t_f) = 0 \qquad \forall\, t \in [t_0, t_f] \,. \quad \checkmark$$

Note: In reality, we showed that $H\big|_* (\cdot) = H(x^*(\cdot), u^*(\cdot), p^*(\cdot))$ is a continuous function, even at corner points where $u(\cdot)$ is potentially not continuous.

Note: We still haven't argued why $H(x^*(t), u, p^*(t))$ is maximized at $u = u^*(t)$. It relates to a geometric cone condition along the curve $\rightarrow$ see (4.29).