

Review:

$$\min_{u(\cdot)} J(u) = \int_{t_0}^{t_f} L(t, x, u) dt + K(t_f, x_{t_f})$$

$$\text{s.t.} \quad \dot{x} = f(t, x, u), \quad x_{t_0} = x_0$$

$$u(t) \in U, \quad \forall t.$$

$$(t_f, x_{t_f}) \in S \subset [t_0, \infty) \times \mathbb{R}^n$$

← dynamics

← input constraint

← target set.

Cost functional:

$$J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t)) dt + K(t_f, x_{t_f})$$

$$L: \mathbb{R} \times \mathbb{R}^n \times U \rightarrow \mathbb{R} \quad \leadsto \text{"running cost"}$$

$$K: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}. \quad \leadsto \text{"terminal cost"}.$$

Note: To relate this back to our VP problem, note that this is more general than the case where $L_{vp} = L_{vp}(t, x, \dot{x})$ because $\dot{x} = f(t, x, u)$ and thus

$$L_{vp} = L_{vp}(t, x, f(t, x, u)) \triangleq L(t, x, u).$$

But, we cannot necessarily go in the opposite direction.

Note: the terminal cost can be turned into a running cost:

$$\begin{aligned}
 K(t_f, x_{t_f}) &= K(t_0, x_{t_0}) + \int_{t_0}^{t_f} \frac{d}{dt} K(t, x_t) dt \\
 &= \underbrace{K(t_0, x_{t_0})}_{\text{constant}} + \int_{t_0}^{t_f} \underbrace{\left[K_t(t, x_t) + K_x(t, x_t) \cdot f(t, x_t, u_t) \right]}_{\triangleq L(t, x, u)} dt
 \end{aligned}$$

Conversely, we introduce a new state variable x^L with

$$\dot{x}^L = L(t, x, u), \quad x^L(t_0) = 0, \quad \text{and then}$$

$$J(u) = \int_{t_0}^{t_f} L(t, x, u) dt = x^L(t_f) \triangleq K\left(t_f, \begin{bmatrix} x \\ x^L \end{bmatrix}(t_f)\right)$$

with the augmented dynamics

$$\frac{d}{dt} \begin{bmatrix} x \\ x^L \end{bmatrix} = \begin{bmatrix} f(t, x, u) \\ L(t, x, u) \end{bmatrix}.$$

Note: we can get rid of time dependence by introducing a new state $x_{n+1} = t \Rightarrow \dot{x}_{n+1} = 1$ and then.

$$J(u) = \int_{t_0}^{t_f} L(x(t), u(t)) dt + K(x_f), \quad x \in \mathbb{R}^{n+1}$$

$$\text{and } \frac{d}{dt} x_t = \begin{bmatrix} f(x(t), u(t)) \\ 1 \end{bmatrix}.$$

Target set:

$\rightarrow x_0, t_0$ are always fixed.

$\rightarrow x_f, t_f$ can be fixed, free, or belong to a set. Generally, we define the "target set" to be a closed subset:

$$S \subset [t_0, \infty) \times \mathbb{R}^n$$

and define t_f as

$$t_f = \inf \{ t : (t, x_t) \in S \}$$

read it as "the first hitting time of S "

→ For now, $t_f = \infty$ is not allowed, but later we will consider these cases called "Infinite-horizon optimal control".

Examples: $S = [t_0, \infty) \times \{x_1\} \rightarrow \begin{cases} t_f \text{ is "free-time"} \\ x_f = x_1 \text{ is "fixed-endpoint"} \end{cases}$

$S = [t_0, \infty) \times M$ ← a manifold/surface

$S = \{t_1\} \times \mathbb{R}^n \rightarrow \begin{cases} t_f = t_1 \text{ is "fixed-time"} \\ x_f \text{ is "free-endpoint"} \end{cases}$

$S = \{t_1\} \times \{x_1\} \rightarrow \text{fixed-time, fixed-endpoint.}$

$S = [t_0, \infty) \times \mathbb{R}^n \rightarrow \text{free-time, free-endpoint}$

(advanced) $S = \{(t, g(t)) : t \in [t_0, \infty)\} \rightarrow \text{moving target set for some continuous } g: \mathbb{R} \rightarrow \mathbb{R}^n$

↳ but for these cases, we can use the trick of introducing $x_{n+1} = t$ and reduce to $\tilde{S} = [t_0, \infty) \times M$

with $M \triangleq \left\{ x \in \mathbb{R}^{n+1} : x = \begin{bmatrix} g(x_{n+1}) \\ x_{n+1} \end{bmatrix} \right\}$.

Fixed-time Free-endpoint optimal control problems

S_0 , $S = \{t_1\} \times \mathbb{R}^n$ for a fixed t_1 , $U = \mathbb{R}^m$ (Unconstrained control)

and $K = K(x_f)$ no time dependence. Then

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt + K(x(t_1))$$

- suppose $u^*(\cdot)$ is a global minimum:

$$J(u^*) \leq J(u), \quad \forall u \text{ piecewise continuous on } [t_0, t_1].$$

- The optimal trajectory is $x^*(\cdot)$ satisfying $\dot{x}^* = f(t, x^*, u^*)$.

- If we try to perturb x^* directly to $x = x^* + \alpha \eta$, then it would be hard to characterize the set of admissible η .

- Instead, we perturb u^* to

$$u^* = u + \alpha \xi, \quad \text{for any } \xi(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^m, \text{ Piecewise-Cont}$$

and this results in a perturbation of x^* as

$$x = x^* + \alpha \eta + o(\alpha).$$

Set of admissible η :

- let $x(t; \alpha)$ be the parameterized trajectory solving

$$\dot{x} = f(t, x, u^* + \alpha \xi), \quad x(t_0) = x_0$$

and, $x(t; \alpha) = x^*(t) + \alpha \eta(t) + o(\alpha)$.

$\Rightarrow$ clearly $\eta(t_0) = 0$ $\because$ $x(t_0) = x_0$ fixed, $x(t; 0) = x^*(t)$.

$\Rightarrow x_\alpha(t; 0) = \eta(t), \quad \forall t.$

$$\frac{d}{dt} \eta(t) = \frac{d}{dt} x_\alpha(t; 0) = x_{\alpha t}(t; 0) = x_{t\alpha}(t; 0) = \left. \frac{d}{d\alpha} \right|_{\alpha=0} x_t(t; \alpha)$$

$$= \left. \frac{d}{d\alpha} \right|_{\alpha=0} f(t, x(t; \alpha), u^* + \alpha \xi)$$

$$= f_x(t, x(t; 0), u^*(t)) x_\alpha(t; 0) + f_u(t, x(t; 0), u^*(t)) \cdot \xi(t)$$

so, $\dot{\eta}(t) = f_x|_* \cdot \eta_t + f_u|_* \cdot \xi(t)$

where $F|_*$ is a useful notation for $F(t, x^*, u^*)$.

$\rightarrow$ note that $f_x|_*$ and $f_u|_*$ are only functions of times:

$$\boxed{\dot{\eta}(t) = A(t) \eta(t) + B(t) \xi(t), \quad \eta(t_0) = 0}$$

admissible perturbations.

→ Think of $\dot{x} = f(t, x, u)$ as a non-Integral constraint and let

$$J(u) = \int_{t_0}^{t_1} \left(L(t, x, u) + p(t) \cdot [\dot{x}(t) - f(t, x(t), u(t))] \right) dt + K(x(t_1))$$

where $p(\cdot) : [t_0, t_1] \rightarrow \mathbb{R}^n$ is a C^1 function.

→ Note that the value of $J(u)$ doesn't change by this as long as $\dot{x}(t) = f(t, x, u)$ is a feasible trajectory!

→ Introduce the Hamiltonian in control context as:

$$H = H(t, x, u, p) \triangleq \underbrace{\langle p, f(t, x, u) \rangle}_{= \dot{x}} - L(t, x, u)$$

- notice the connection to previous Hamiltonian where u replaces y' .

Therefore,

$$J(u) = \int_{t_0}^{t_1} \left[\langle p(t), \dot{x}(t) \rangle - H(t, x(t), u(t), p(t)) \right] dt + K(x(t_1))$$

Q: What is the first variation $\delta J|_{u^*}[\xi]$?

$$J(u) = J(u^*) + \delta J|_{u^*}[\xi] \cdot \alpha + o(\alpha).$$

$$J(u) - J(u^*) = \underbrace{K(x(t_1)) - K(x^*(t_1))}_{\text{I}} + \underbrace{\int_{t_0}^{t_1} \langle p(t), \dot{x}(t) - \dot{x}^*(t) \rangle dt}_{\text{II}} - \underbrace{\int_{t_0}^{t_1} [H(t, x(t), u(t), p(t)) - H(t, x^*(t), u^*(t), p(t))] dt}_{\text{III}}.$$

$$\begin{aligned} \text{I} &= K(x^*(t_1) + \alpha \eta(t_1) + o(\alpha)) - K(x^*(t_1)) \\ &= \langle K_x(x^*(t_1)), \alpha \eta(t_1) \rangle + o(\alpha). \end{aligned}$$

$$\begin{aligned} \text{II} &= \langle p(t), x(t) - x^*(t) \rangle \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \langle \dot{p}(t), x(t) - x^*(t) \rangle dt \\ \text{(Integration by part)} &= \langle p(t_1), \alpha \eta(t_1) \rangle - \int_{t_0}^{t_1} \langle \dot{p}(t), \alpha \eta(t) \rangle dt + o(\alpha) \end{aligned}$$

$$\begin{aligned} \text{III} &= \int_{t_0}^{t_1} [H(t, x^*(t) + \alpha \eta(t) + o(\alpha), u^*(t) + \alpha \xi(t), p(t)) - H(t, x^*, u^*, p)] dt \\ &= \int_{t_0}^{t_1} [\langle H_x|_{x^*}, \alpha \eta(t) \rangle + \langle H_u|_{u^*}, \alpha \xi(t) \rangle] dt + o(\alpha). \end{aligned}$$

therefore,

$$\begin{aligned} \delta J|_{u^*}(\xi) &= - \int_{t_0}^{t_1} [\langle \dot{p} + H_x(t, x^*, u^*, p), \eta \rangle + \langle H_u(t, x^*, u^*, p), \xi \rangle] dt \\ &\quad + \langle K_x(x^*(t_1)) + p(t_1), \eta(t_1) \rangle \end{aligned}$$

where $\bar{\eta}_t = f_x|_{\star} \cdot \eta_t + f_u|_{\star} \cdot \xi_t$, $\eta(t_0) = 0$.

First-order nec. condition:

$\delta J|_{u^*}[\xi] = 0$, $\forall \xi(\cdot)$ admissible and all $p_i \in \mathbb{C}^1$.

choose: $\dot{p} = -H_x(t, x^*, u^*, p)$ and $p(t_1) = -K_x(x^*(t_1))$
and let's denote its solution by $p^*(t)$.

Then: $\delta J|_{u^*}[\xi] = - \int_{t_0}^{t_1} \langle H_u|_{\star}, \xi \rangle dt = 0$, $\forall \xi(\cdot)$ admissible

by 1st-fundamental
lemma of cal. of var.

$\Rightarrow H_u(t, x^*, u^*, p^*) = 0 \quad \forall t \in [t_0, t_1]$.

$\rightarrow$ this means $H(t, x^*, \cdot, p^*)$ has a stationary point at u^* .

Finally: Recall that $\dot{x}^* = H_p|_{\star}$ and $x(t_0) = x_0$

let's look at $\dot{p}^* = -H_x|_{\star}$ more closely:

$$\begin{aligned} \dot{p}^* &= -H_x(t, x^*, u^*, p^*) = -\langle p^*(t), f_x|_{\star} \rangle + L_x|_{\star} \\ &= -\left(f_x|_{\star}\right)^T p^*(t) + L_x|_{\star} \\ &= -A_{\star}^T(t) p^* + L_x|_{\star} \end{aligned}$$

For example, if $\dot{x} = Ax + Bu$ then $f_x = A$ and then

$$\dot{p}^* = -A^T p^* + L_x|_* \quad \rightarrow \text{called the "adjoint dynamics"}$$

and p is called the "costate".

So far:

$$\begin{aligned} \dot{x}^* &= H_p|_* , & x^*(t_0) &= x_0 \\ \dot{p}^* &= -H_x|_* , & p^*(t_1) &= -K_x(x^*(t_1)) \\ H_u|_* &= 0 , & \forall t &\in [t_0, t_1] \end{aligned}$$

← state dynamics

← costate adjoint dyn.

← Hamiltonian stationarity

Note: the Hamiltonian here is defined as $H = H(t, x, \overset{\uparrow}{u}, p)$
"not $\dot{x}$ " .

To clarify the connection to the Lagrangian and the Lagrange multipliers: Complete Exercises 3.5, 3.6.

Second variations:

Consider a similar perturbation $u = u^* + \alpha \xi$, but now look at the second order perturbation on x :

$$x = x^* + \alpha \eta + \alpha^2 \zeta + o(\alpha^2) \quad \text{for some } \eta(\cdot), \zeta(\cdot).$$

. 11.4

Exercise: By a similar analysis show now

$$\delta^2 J|_{u^*}[\xi] = -\frac{1}{2} \int_{t_0}^{t_1} \begin{pmatrix} \eta \\ \xi \end{pmatrix}^T \begin{pmatrix} H_{xx} & H_{xu} \\ H_{xu} & H_{uu} \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} dt + \frac{1}{2} \eta^T K_{xx}(x^*(t_1)) \eta$$

where $\dot{\eta} = (f_x|_{u^*}) \eta + (f_u|_{u^*}) \xi$, $\eta(t_0) = 0$.

(Hint: Note that we have already the 1st order nec. conditions.)

Now, the 2nd-order nec. condition reads:

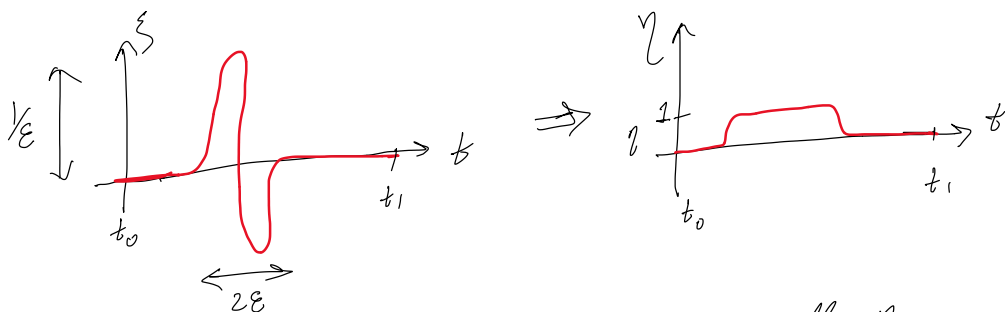
$$\delta^2 J|_{u^*}[\xi] \geq 0, \quad \forall \xi \text{ admissible.}$$

→ What can we conclude from this?

- Recall that we have asked a similar question before when we wanted to simplify $\int_a^b [p(\eta')^2 + Q(\eta)^2] dt \geq 0, \quad \forall \eta \text{ admissible.}$

resulting in $P(t) \geq 0$ p.s.d. $\forall t \in [a, b]$.

- Here, $\dot{\eta} = A_* \eta + B_* \xi$ which means (again):



so, large $\xi \Rightarrow$ small η .

therefore, the term of $\delta^2 J|_{u^*}(\xi)$ that is depends on ξ with highest degree dictates the sign. Thus,

$$\delta^2 J|_{u^*}(\xi) \geq 0, \forall \xi \text{ admissible}$$

$$\Rightarrow \xi^T H_{uu}|_{u^*} \xi \leq 0, \forall \text{ such large such } \xi \text{ that result in small } \eta.$$

$$\Rightarrow \boxed{H_{uu}(t, x^*(t), u^*(t), p^*(t)) \leq 0 \text{ N.S.D.}, \forall t \in [t_0, t_1]}$$

called "Legendre-Clebsch" condition

But, recall that $H_u(t, x^*, u^*, p^*) = 0$

$$\Rightarrow \underline{H(t, x^*, \cdot, p^*) \text{ has a stationary point at } u^*}.$$

therefore, it is tempting to conjecture that:

$$\underline{H(t, x^*, \cdot, p^*) \text{ obtains its local maximum at } u^*!}$$

This conjecture is true! and in summary we have

necessary conditions for optimality:

If $u^*(\cdot)$ is an optimal control, with corresponding trajectory $x^*(\cdot)$

then $\exists p^*(\cdot)$ costate/adjoint vector s.t.

$$\begin{cases} \dot{x}^* = H_p|_* & , x(t_0) = x_0 \\ \dot{p}^* = -H_x|_* & , p(t_1) = -K_x(x(t_1)) \end{cases}$$

and for each fixed $t \in [t_0, t_1]$, the function

$$u \mapsto H(t, x^*(t), u, p^*(t))$$

has a local maximum at $u = u^*(t)$: i.e., $\forall t \in [t_0, t_1]$,

$$H(t, x^*(t), u^*(t), p^*(t)) \geq H(t, x^*(t), u, p^*(t)) \quad , \quad \forall u \text{ near } u^*(t).$$

Fun Exercises :

1) Show that in the case of directly controllable system
 $\dot{x} = u$, the costate will reduce to the momentum
 that we defined in the variational problem.

2) Show that in general we have :

$$\boxed{\frac{d}{dt} H|_* = H_t|_*}$$

and, in particular, this means for time-invariant optimal control problems, the Hamiltonian remains constant on

$$\text{the optimal trajectory : } \frac{d}{dt} H|_* = 0 ,$$

$\Rightarrow$ the "energy" is conserved !

Important Exercise: solve Exercise 3.8.

In particular, argue that your solution satisfies the

sufficient condition $S^2 J|_{u^*}(\xi) > 0, \forall \xi \neq 0,$

and thus you have indeed characterized the optimal solution.