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Online Regulation of Unstable Linear Systems from a Single Trajectory

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Abstract—**In this paper, we study online regulation of partially unknown (and possibly unstable) linear systems. In order to avoid the popular assumption of having access to an initial stabilizing controllers in learning algorithms, we propose the Data-Guided Regulator (DGR) synthesis that regulates the underlying states of an unknown linear model through generating informative data. We also introduce the notion of "regularizability" for a linear system that is of independent interest and provides a unique perspective on the geometry of data-guided regulation. Finally, we discuss an example involving online regulation of the (open-loop unstable) X-29 aircraft.**

Index Terms*— *Online Regulation; Unstable Linear Systems; Single-Trajectory Learning; Iterative Control

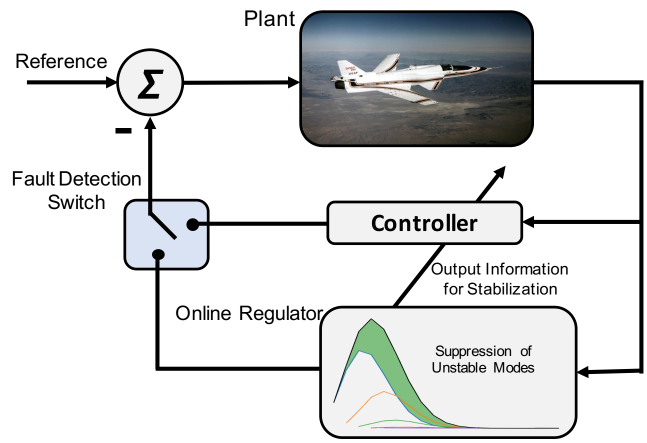

Fig. 1: Data-guided online regulation

I. Introduction

Feedback synthesis has recently been approached from the perspective of data-guided control. This trend has been encouraged by modern sensing technologies and efficient computational methods, where the system designer aims to reason about control and estimation of uncertain systems- all from measured data. This perspective is particularly important for safety-critical systems, necessitating non-asymptotic analysis on data-driven methods. In particular, there has been a growing interest in finite-time control and analysis of unknown linear dynamical systems from time-series or a single trajectory [1]–[4]. Parallel to asymptotic analysis in traditional adaptive control and system identification [5], a model-based non-asymptotic control analysis has benefited from a least-squares approach to identification followed by robust synthesis [6]. It can be shown that under specific assumptions, model-based methods may require fewer measurements; however, for certain control problems data collection can be still expensive or impractical due to resource limitations and safety constraints. Furthermore, many of these studies rely on *a priori* information about the system, such as estimates of system parameters [7], an initial stabilizing controller [8], or a stable open-loop system [4], [9]. In addition, persistently exciting inputs are not practical for control and identification of unstable systems even in low dimensions without recourse to resets [10].[1]

The overarching thesis for the present work revolves around the realization that all variants of data-guided methods for control encounter a major challenge when faced with unstable (unknown) systems. This includes "offline" realizations of data-guided control such as those relying on system identification or batch processing for control synthesis, as well as those that are based on adaptation and reinforcement learning. This state of affairs is partially due to the fact that unstable systems are fundamentally more difficult to control as elaborated upon eloquently by Stein [11]; in fact, practical closed-loop systems with unstable subsystems are only locally stable [12]. A "data-guided" perspective only alleviates an already challenging synthesis process.

In this paper, we introduce the Data-Guided Regulator (DGR), an online iterative synthesis procedure that utilizes a single trajectory for an otherwise unknown discrete-time linear (time-invariant) system (see Figure 1). In particular, DGR does not use prior assumptions on neither the structure of state coupling dynamics nor the stability property of the initial controller; instead, the algorithm only utilizes the knowledge of the input matrix. The knowledge of the input matrix is motivated by scenarios where it is at least known how the control input effects the state dynamics, yet how the internal states of the system interact amongst each other is uncertain.[2]

The contribution of the proposed work is threefold: (a) We show conditions under which DGR can eliminate unstable modes of the (unknown) system and regulate (a notion that we will subsequently clarify) the state trajectory. As such, DGR can avoid the ill-posedness of the data generation process for unstable systems; (b) Our analysis

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[1] For instance, injecting white noise to an unstable system can result in ill-conditioned data matrices even for a few number of states.

[2] We postulate that a general online data-guided stabilization process for unstable systems without knowledge of neither system nor input matrices, at least intuitively, is a tall order. In this case, a batch processing of input-output data–when feasible–can be adopted.

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underscores that conventional system theoretic notions such as controllability *do not* capture online "regularizability" of a system. As such, one needs to characterize a notion that we refer to as regularizability; this construct connects well with other fundamental concepts in linear system theory, yet highlights unique geometric aspects of *online* data-guided synthesis. We then proceed to derive upper bounds on the state trajectories based on another geometric property of the system that we refer to as its "instability number;" (c) Finally, we show that while DGR performs well for a large class of unstable systems, special system structures (e.g., symmetry) facilitates further analysis for deriving intuitive bounds on the system trajectory.

The rest of the paper is organized as follows. In §II, we introduce the problem setup as well as a motivating example, followed by introducing the DGR algorithm. The analysis of the corresponding closed loop system and upper bounds on the state trajectories are discussed in §III. We provide an example in §IV followed by concluding remarks in §V.[3]

Notation: The unit vector $\boldsymbol{e}_i$ is a column vector with identity in the i-th entry and zero elsewhere. The range and nullspace of a real matrix $M \in \mathbb{R}^{n\times m}$ are denoted by $\mathcal{R}(M) \subseteq \mathbb{R}^n$ and $\mathcal{N}(M) \subseteq \mathbb{R}^m$, respectively. The span of a set of vectors over the complex field is denoted by $\mathrm{span}\{.\}$. The Singular Value Decomposition (SVD), thin SVD and Moore–Penrose generalized inverse are adopted from [14], where the latter is denoted by $(.)^\dagger$. A square matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable if $\rho(A) < 1$, where $\rho(.)$ denotes the spectral radius. Additionally, A is (complex) *diagonalizable* if there exist a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $U \in \mathbb{C}^{n\times n}$, such that $A = U\Lambda U^{-1}$. The orthogonal projection of a vector $\boldsymbol{v}$ on a linear subspace S is denoted by $\mathrm{Proj}_S(\boldsymbol{v})$. The *Euclidean norm* of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by $\|\boldsymbol{x}\| = (\boldsymbol{x}^\top \boldsymbol{x})^{1/2}$. For a matrix M, its *operator norm* is also denoted by $\|M\| = \sup\{\|M\boldsymbol{u}\| : \|\boldsymbol{u}\| = 1\}$. We say that $\boldsymbol{x}_0$ *excites* k *modes of a diagonalizable matrix* A if $\boldsymbol{x}_0$ is contained in the (complex-)span of k eigenvectors of A, but not in the span of any $k-1$ eigenvectors; we refer to those k eigenvectors (needed for forming $\boldsymbol{x}_0$) as the *excited modes*.

II. Problem Setup

In this section, we introduce the problem formulation and highlight some of its unique features through an example. Consider a discrete-time LTI system of the form,

$$\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + B\boldsymbol{u}_t, \qquad \boldsymbol{x}_0 \ \text{given}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system parameters and $\boldsymbol{x}_t \in \mathbb{R}^n$ and $\boldsymbol{u}_t \in \mathbb{R}^m$ denote the state vector and control inputs at iteration t, respectively. We assume that the system matrix A is unknown and (possibly) unstable, and that the input matrix B is known. The problem of interest is to design $\boldsymbol{u}_t$ from state measurements such that: I) the system is regulated and with a state norm that is uniformly bounded during the learning process, e.g., $\boldsymbol{x}(t)$ evolves in a safe region ($\|\boldsymbol{x}_t\|$ remains "bounded" in a quantifiable manner) and the corresponding data matrix does not become ill-conditioned, and II) the system generates informative data for post-processing such as data-driven stabilization or system identification.[4]

Considering online regulation given the input matrix is of great interest in certain applications [15], where it is known a priori how various control inputs affect the dynamic states, e.g., the effect of elevator deflection on the pitch dynamics for longitudinal aircraft dynamics. Intuitively, this assumption allows online regulation to even have a chance of stabilizing an unknown unstable system in real-time. The following example further motivates our setup and underscores why the data-guided perspective requires introducing new system theoretic constructs.

Example 1. *For any positive integer n, define the system matrix $A = [A_{i,j}] \in \mathbb{R}^{n\times n}$ and the input matrix $B = \boldsymbol{e}_n \in \mathbb{R}^n$ where $A_{i,i} = \lambda_i$ and $A_{i,i+1} = 1$ for $i = 1, \dots, n-1$, $A_{n,n} = \lambda_n$ and otherwise $A_{i,j} = 0$. Note that for any choice of $\lambda_i \in \mathbb{R}$, the pair (A, B) is controllable (and therefore stabilizable). Let $\boldsymbol{x}_0 = \boldsymbol{e}_1$ and observe that under* (1)*, we have $\mathbf{e}_1^\top \boldsymbol{x}_t = \lambda_1^t$ for all $0 \le t < n$ regardless of the input signal $\boldsymbol{u}_t$. This implies that, for "any" choice of input, for the first $(n-1)$ iterations, the first state of the system grows exponentially fast with the rate λ_1 whenever $|\lambda_1| > 1$.*

Remark 2. *Example 1 constructs a family of controllable systems where no controller can regulate their respective first states. This example highlights that controllability of a pair (A, B) does not capture online regularizability of a linear system, when the regulation has been achieved in a data-guided manner and the controller does not have access to enough samples.*

Data-Guided Regulator (DGR) Algorithm: The primary focus of the present work is devising an online, data-driven feedback controller to regulate the system state trajectories, quantified in terms of some signal norm. That is given a streaming (closed loop) data set $\mathcal{D}_t$ available at time t, we are interested in characterizing the algorithmic map $\mathcal{D}_t \to \mathcal{K}$, where $\mathcal{K}$ is a set of regulating state feedback gains for the unknown dynamics. In fact, we propose an iterative algorithm for updating the feedback gain (policy) based on the available (closed loop) data; the form of the controller is "motivated" by considering, at each time-step t, minimizing $\|\boldsymbol{x}_{t+1}\|^2$, subject to (1).[5]

In the case of known A, it is straightforward to characterize the set of minimizers of the above optimization problem through the first order optimality condition as,

$$B^\top B\boldsymbol{u}_t + B^\top A\boldsymbol{x}_t = 0;$$

as such the corresponding input belongs to a linear subspace in $\mathbb{R}^m$ parameterized by the system matrices and data. This

[3]See [13] for an extended version of this work.

[4]We interchangeably use the terms linear independence and *informativity* of data to emphasize that the collected data has useful information content for decision-making.

[5]A more elaborate form of the optimization problem is assumed in [13], where it also includes a penalty on the input.

Algorithm 1 Data-Guided Regulator (DGR)

Initialization (at $t = 0$)
  Measure $\boldsymbol{x}_0$; set $K_0 = 0$ and $G = (B^\top B)^\dagger B^\top$
  Set $\mathcal{X}_0 = [\boldsymbol{x}_0]$ and $\mathcal{Y}_0 = [\ ]$
while stopping criterion not met[6]
  Compute $\boldsymbol{u}_t = -K_t \boldsymbol{x}_t$
  Run system (1) and measure $\boldsymbol{x}_{t+1}$
  Update $\mathcal{Y}_{t+1} = \begin{bmatrix} \mathcal{Y}_t & \boldsymbol{x}_{t+1} - B\boldsymbol{u}_t \end{bmatrix}$
    $K_{t+1} = G\mathcal{Y}_{t+1}\mathcal{X}_t^\dagger$
    $\mathcal{X}_{t+1} = \begin{bmatrix} \mathcal{X}_t & \boldsymbol{x}_{t+1} \end{bmatrix}$
  $t = t + 1$

observation forms the basis for the proposed algorithm in this paper when A is unknown and potentially unstable; essentially, we aim to control the system in an online manner towards a more stable regime, using the history of the state data and a minimum norm input. The corresponding synthesis procedure is detailed in Algorithm 1. Specifically, at time-step t, DGR sets

$$\boldsymbol{u}_t^* = -K_t^* \boldsymbol{x}_t, \quad K_t^* := (B^\top B)^\dagger B^\top \mathcal{Y}_t \mathcal{X}_{t-1}^\dagger, \tag{2}$$

where $\mathcal{X}_{t-1}, \mathcal{Y}_t \in \mathbb{R}^{n \times t}$ are the measured data matrices with $\mathcal{Y}_t = A\mathcal{X}_{t-1}$. Intuitively, collecting more data results in capturing the essential (e.g., unstable) modes in the dynamics. As such, it is important to note that DGR is particularly relevant for online regulation of an unstable system, when the controller does not have access to enough data for the purpose of system identification or optimal control synthesis.

Finally, note that DGR actively guides the ongoing process, and the data generation process and the dynamics are interdependent. The proposed technique is close in spirit to modal analysis where regression-based methods are leveraged to extract and control the dominant modes of the system [17]. The emphasis of DGR, however, is on the significance of each temporal action for safety-critical applications; in these scenarios, it might be rather unrealistic to generate enough data from the inherent unstable modes.

III. Main Results

We provide our analysis in two subsections. First, we present the results for the general case where $B \in \mathbb{R}^{n \times m}$ is an arbitrary input matrix. Then to further examine the effects of DGR on the system state trajectory, we focus on the restricted case where $\mathcal{R}(A) \subset \mathcal{R}(B)$. In addition, we will see how a particular structure of the system matrix A, such as diagonalizability and symmetry, would facilitate more insights into the operation of DGR.

A. Case 1: General Input Matrix B

In this section, we provide necessary conditions for achieving the desired performance of DGR, particularly when $t \leq n$. To begin, we first define a novel system property that captures the effectiveness of the input characteristics pertinent to online regulation; this property provides the basis for our subsequent analysis of DGR.

Definition 3. *We say that the pair (A, B) is "regularizable" if $\mathrm{Proj}_{\mathcal{R}(B)^\perp} A$ is Schur stable.*

As we will show subsequently, the regularizability of a pair (A, B) is related to the detectability of $(A, B^\top)$; a combination that traditionally has not been examined. This connection is intuitive, as regulation of a system in finite-time requires the states to be accessible through the input matrix B. However, before delving into this further, we show why regularizability is essential in the analysis of the trajectory generated under Algorithm 1 for DGR.

Lemma 4. *The trajectory generated by Algorithm 1 satisfies,*

$$\boldsymbol{x}_{t+1} = \mathrm{Proj}_{\mathcal{R}(B)^\perp} A\boldsymbol{x}_t + \mathrm{Proj}_{\mathcal{R}(B)}\, A\boldsymbol{z}_t,$$

where $\boldsymbol{z}_0 := \boldsymbol{x}_0$ and $\boldsymbol{z}_t := \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{t-1})^\perp} \boldsymbol{x}_t$ for $t > 0$. Furthermore, $\{\boldsymbol{z}_0, \boldsymbol{z}_1, \cdots, \boldsymbol{z}_t\}$ is a set of orthogonal vectors.

Proof. Let $B = U_r \Sigma_r V_r^\top$ be the "thin" SVD of B where $r = \mathrm{rank}(B)$. Since $B(B^\top B)^\dagger B^\top = U_r U_r^\top$,

$$\begin{aligned}
\boldsymbol{x}_{t+1} &= \left[A - U_r U_r^\top \mathcal{Y}_t \mathcal{X}_{t-1}^\dagger\right]\boldsymbol{x}_t \\
&= \left[A - \mathrm{Proj}_{\mathcal{R}(U_r)}\, A\, \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{t-1})}\right]\boldsymbol{x}_t \\
&= \mathrm{Proj}_{\mathcal{R}(U_r)^\perp} A\boldsymbol{x}_t + \mathrm{Proj}_{\mathcal{R}(U_r)}\, A\, \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{t-1})^\perp} \boldsymbol{x}_t \\
&= \mathrm{Proj}_{\mathcal{R}(U_r)^\perp} A\boldsymbol{x}_t + \mathrm{Proj}_{\mathcal{R}(U_r)}\, A\boldsymbol{z}_t.
\end{aligned}$$

Thus, the first claim follows since $\mathcal{R}(U_r) = \mathcal{R}(B)$. For the second claim, note that the definition of $\boldsymbol{z}_t$ implies $\boldsymbol{z}_t \perp \mathcal{R}(\mathcal{X}_{t-1})$ and $\boldsymbol{z}_k \in \mathcal{R}(\mathcal{X}_{t-1})$, for all $k = 0, 1, \cdots, t-1$. Hence $\{\boldsymbol{z}_0, \boldsymbol{z}_1, \cdots, \boldsymbol{z}_t\}$ consists of orthogonal vectors. ☐

The preceding lemma implies that the trajectory generated by Algorithm 1, can be considered as the trajectory of a linear system with parameters $(\widetilde{A}, \widetilde{B})$ and input $\boldsymbol{z}_t$ where,

$$\widetilde{A} := \mathrm{Proj}_{\mathcal{R}(B)^\perp} A, \qquad \widetilde{B} := \mathrm{Proj}_{\mathcal{R}(B)} A, \tag{3}$$

and $\boldsymbol{z}_t = \widetilde{K}_t \boldsymbol{x}_t$, with the time-varying, state-dependent feedback gain $\widetilde{K}_t = \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{t-1})^\perp}$. Equivalently,

$$\boldsymbol{x}_{t+1} = \widetilde{A}^{t+1}\boldsymbol{x}_0 + \sum_{r=0}^{t} \widetilde{A}^{t-r}\widetilde{B}\boldsymbol{z}_r. \tag{4}$$

Note that if $\mathrm{Proj}_{\mathcal{R}(B)}$ and A commute,[7] then $\widetilde{A}\widetilde{B} = 0$ and $\boldsymbol{x}_{t+1} = \widetilde{A}^{t+1}\boldsymbol{x}_0 + \widetilde{B}\boldsymbol{z}_t$. Moreover, $\widetilde{A} = 0$ whenever $\mathcal{R}(A) \subset \mathcal{R}(B)$, i.e., the system dynamics will be driven only by the feedback signal $\boldsymbol{z}_t$; these cases will be examined further subsequently. Finally, note that an attractive feature of DGR hinges upon the orthogonality of the "hidden" states $\boldsymbol{z}_t$ generated during the process.

Remark 5. *In a nutshell, DGR regulates the unknown system (when regularizable) by sequentially detecting its unstable behavior and exploiting this information for input synthesis.*

[6]The stopping criterion can be application specific. For instance, generating n linearly independent data is sufficient for system identification (sysID), while mere stabilization may require less; see [16].

[7]This is the case if (and only if) both matrices are simultaneously diagonalizable (see Theorem 1.3.21 in [14]).

1) Regularizable Systems: In order to get a better sense of the notion of regularizability, we study the spectral properties of $\widetilde{A}$ in (3) and its relation with the original matrices A and B. First, the following example highlights why regularizability of a system is distinct from its controllability.

Example 6. *Consider the system with the A matrix defined as in Example 1, where $|\lambda_1| > 1$ and $|\lambda_i| < 1$ for $i = 2, \ldots, n$. Note that the pair $(A, \boldsymbol{e}_n)$ is controllable (and thus stabilizable), but not regularizable. On the other hand, the pair $(A, \boldsymbol{e}_1)$ is regularizable but not controllable.*

Recall that a pair (A, B) is stabilizable if and only if $(A^\top, B^\top)$ is detectable [18]; the detectability of $(A, B^\top)$ is seldom of interest in linear system theory. However, we show that detectability of the pair $(A, B^\top)$ is, indeed, a necessary condition for (A, B) to be regularizable.

Proposition 7. *If (A, B) is regularizable, then $(A, B^\top)$ is detectable.*

Proof. We prove the contrapositive. Suppose that $(A, B^\top)$ is not detectable. Hence, there exists a right eigenpair $(\lambda, \boldsymbol{v})$ of A, where $|\lambda| \geq 1$ and $\boldsymbol{v} \in \mathcal{N}(B^\top) = \mathcal{R}(B)^\perp$. Then, directly by definitions we infer that $(\lambda, \boldsymbol{v})$ must be a right eigenpair of $\tilde{A}$. Since $|\lambda| \geq 1$, $\tilde{A}$ is not Schur stable and therefore (A, B) is not regularizable. $\square$

Note that Proposition 7 provides a necessary condition for regularizability. To show that the detectability of $(A, B^\top)$ is not sufficient for regularizability, we will construct a counterexample through the pair (A_1, B_1). Such a pair is constructed as follows. Let A be as in Example 6 and observe that $(A^\top, \boldsymbol{e}_2^\top)$ is detectable. However, the elements of $\mathrm{Proj}_{\mathcal{R}(B)^\perp} A^\top$ are equal to those of $A^\top$ except its second column which are all zeros. Since $\lambda_1 > 1$, we conclude that $\mathrm{Proj}_{\mathcal{R}(B)^\perp} A^\top$ is not Schur stable, which in turn, implies that $(A^\top, \boldsymbol{e}_2)$ is not regularizable. Now set $A_1 = A^\top$ and $B_1 = \boldsymbol{e}_2$. Note that Proposition 7 equivalently states that the pair $(A^\top, B)$ is stabilizable whenever (A, B) is regularizable. That is, for the system in (3) to be Schur stable, the transposed system has to be stabilizable through B.[8]

2) Bounds on the State Trajectories: In an open loop setting, the generated data from an unstable system can grow exponentially fast with a rate dictated by the largest unstable mode. We show that the proposed DGR can prevent this undesirable phenomenon for unstable systems when the system is regularizable. The key property for such an analysis involves the notion of "instability number."

Definition 8. *Given the matrix $A \in \mathbb{R}^{n\times n}$, its instability number of order t is defined as,*

$$M_t(A) \coloneqq \sup_{\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_t\}\in\mathcal{O}_t^n} \|A\boldsymbol{v}_1\|\,\|A\boldsymbol{v}_2\| \;\cdots\; \|A\boldsymbol{v}_t\|, \quad (5)$$

where $\mathcal{O}_t^n$ is the collection of all sets of t orthonormal vectors in $\mathbb{R}^n$.

Note that $M_t(A) \leq \|A\|^t$, where $\|.\|$ denotes the induced operator norm. In general, the instability number of a matrix is distinct from products of any subset of its eigenvalues. Consider for example, a t-dimensional hypercube in the domain with its image as a parallelotope. The instability number is related to the multiplication of the length of edges radiating from one vertex of the parallelotope, while $\det(A^\top A)$ (the multiplication of eigenvalues) is related to its volume. In fact, the instability number of a matrix might be difficult to compute. In what follows, we first provide upper and lower bounds on $M_t(A)$ characterizing its growth rate with respect to the largest singular value of A (where the proof is skipped for brevity). Subsequently, these bounds will be used to provide a bound on the norm of the state trajectory generated by DGR.

Lemma 9. *Let $\sigma_1, \cdots, \sigma_n$ denote the singular values of $A \in \mathbb{R}^{n\times n}$ in descending order. Then for $t \leq n$,*

$$\left[\frac{\sigma_1^2}{t}\right]^t \leq M_t^2(A) \leq \sum_{j=0}^{t-1} \left[\frac{\sigma_1^2}{t-j}\right]^{t-j} \binom{t}{j} \delta^j + \delta^t,$$

where $\delta \coloneqq \sum_{i=2}^t \sigma_i^2$, with $M_t(A)$ as in Definition 8.

The lower and upper bounds in Lemma 9 show that, when $\delta < 1$, $M_t(A)$ would initially grow similar to $(\sigma_1/\sqrt{k})^k$ for $k \leq t$, in contrast to the exponential growth σ_1^k. This fact is illustrated via an example in Figure 2 where the first 5 terms of the upper-bounds in the sum are plotted, and the green region shows where the actual value of $M_t^2(A)$ lies.

Next, we provide an upperbound for the most general case whereafter the connection to the instability number of the system is explained.

Theorem 10. *For any pair (A, B), the trajectory generated by Algorithm 1 satisfies,*

$$\|\boldsymbol{x}_{t+1}\| \leq L_{t+1}\|\boldsymbol{x}_0\|\,,$$

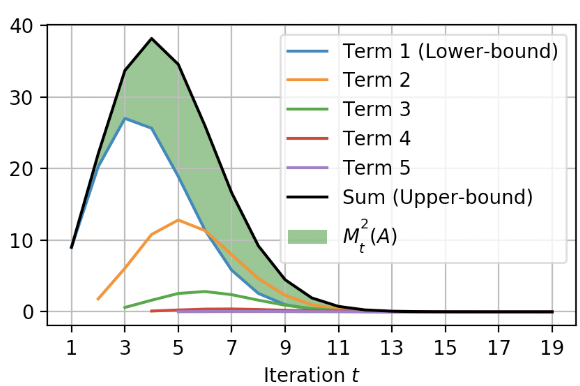

Fig. 2: Illustration of the upper and lower bounds for instability number of a system with $\sigma_1 = 3$ and $\delta = 0.1$ as in lemma 9.

[8]However, the converse is not true in general and the full characterization of regularizability is more elaborate. The reader is referred to [13] for more details.

where L_t satisfies the recursion $L_1 = \|A\overline{\boldsymbol{z}}_0\|$, $L_{t+1} = a_t + \sum_{r=1}^{t} b_{t,r} L_r$, with $a_t = \|\widetilde{A}^t A\overline{\boldsymbol{z}}_0\|$, $b_{t,r} = \|\widetilde{A}^{t-r}\widetilde{B}\overline{\boldsymbol{z}}_r\|$, and $\overline{\boldsymbol{z}}_r = \boldsymbol{z}_r/\|\boldsymbol{z}_r\|$ (if $\boldsymbol{z}_r \neq 0$, otherwise $\overline{\boldsymbol{z}}_r = 0$).

Proof. Knowing that $\boldsymbol{x}_1 = A\boldsymbol{x}_0$ it follows that $\|\boldsymbol{x}_1\| \leq L_1\|\boldsymbol{x}_0\|$. Furthermore, for $t \geq 1$, (4) leads to, $\boldsymbol{x}_{t+1} = \widetilde{A}^t A\boldsymbol{x}_0 + \sum_{r=1}^{t} \widetilde{A}^{t-r}\widetilde{B}\boldsymbol{z}_r$, since $\widetilde{A} + \widetilde{B} = A$. This implies that $\|\boldsymbol{x}_{t+1}\| \leq a_t\|\boldsymbol{x}_0\| + \sum_{r=1}^{t} b_{t,r}\|\boldsymbol{x}_r\|$, where we have used $\|\boldsymbol{z}_r\| \leq \|\boldsymbol{x}_r\|$ as the orthogonal projection is non-expansive. Using this recursive bound, the rest of the proof follows by induction. □

Remark 11. *Note that in the analysis above, when the system is regularizable, a_t decreases exponentially fast as t increases. Furthermore, for each fixed t, the sequence of terms $b_{t,r}$ in the sum is increasing in r. Additionally, since $b_{r,r} = \|\widetilde{B}\overline{\boldsymbol{z}}_r\| \leq \|A\overline{\boldsymbol{z}}_r\|$, for any set $T = \{r_i\}_1^n \subset \mathbb{N}$, $b_{r_n,r_n} b_{r_{n-1},r_{n-1}} \cdots\ b_{r_1,r_1} \leq M_n(A)$. Finally, we can show that the obtained upperbound is tight by considering Example 1 with $\lambda_1 > 0$ and $\lambda_i = 0$ for $i > 1$.*

In order to shed light on the meaning of the upper bound in Theorem 10, we next study simpler cases where there exists small enough κ for which $b_{t,r} \leq \|\widetilde{A}^{t-r}\widetilde{B}\| \leq \kappa$ for all $r < t$. In particular, we can show that if the system is regularizable and $\widetilde{A}\widetilde{B} = 0$, then the trajectories of the closed loop system will be bounded by a combination of instability number of different orders. This is stated in the next corollary; the proof is omitted for brevity.

Corollary 12. *For any regularizable pair (A, B) with $\tilde{A}\tilde{B} = 0$, and $M_t(A)$ as in Definition 8,*

$$\frac{\|\boldsymbol{x}_{t+1}\|}{\|\boldsymbol{x}_0\|} \leq M_{t+1}(A) + a_t + \sum_{r=1}^{t-1} M_r(A)a_{t-r}\,.$$

The above observation further highlights the importance of the instability number in the context of DGR.

B. Case 2: $\mathcal{R}(A) \subset \mathcal{R}(B)$

In order to better understand the behavior of DGR, in this section, we study the more restricted case where $\mathcal{R}(A) \subset \mathcal{R}(B)$. This also includes the case where $\mathrm{rank}(B) = n$, i.e., one can directly control each state of the system (e.g. see [19]). Note that $\mathcal{R}(A) \subset \mathcal{R}(B)$ implies that $\widetilde{A} = 0$ which, in turn, results in regularizability of (A, B). In the sequel, we first establish the linear independence of the data produced by DGR and then make a connection between this independence and the number of excited modes in the system. Furthermore, we show how the simplified bounds (derived in Section III-A) clarify the elimination of the unstable modes in the system.

Before we proceed with presenting the main results of this section, let us define the Vandermonde matrix $L_k^t(A)$ that is based on k eigenvalues of the diagonalizable matrix A as,

$$L_k^t(A) := \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{t-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{t-1} \end{pmatrix}, \quad 1 \leq t \leq n. \quad (6)$$

Note that informativity of the data implies that the data is sufficient for regulating the state trajectory of the system initiated from $\boldsymbol{x}_0$. In particular, if $\boldsymbol{x}_0$ results in all the modes of A being excited, one should be able to conclude that the generated data is informative for regulation. This can be formalized as follows where the proof has been removed due to space constraint.

Proposition 13. *Suppose A is diagonalizable with $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and let $\boldsymbol{x}_0$ excite k modes of A. If $\mathrm{rank}\big(L_k^r(A)\big) = r$ for any $r \leq k$, then $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\}$ is a set of linearly independent vectors.*

An immediate consequence of the above result is that DGR generates data that is effective for simultaneous identification of modes with multiplicity greater than one.

Theorem 14. *Suppose that A is diagonalizable with $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, and let $\boldsymbol{x}_0$ excite k modes of A corresponding to r distinct eigenvalues (where possibly $r \leq k$). Then, in r iterations, $\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\}$ coincides with the subspace containing the excited modes; furthermore, $\boldsymbol{x}_{r+1} = 0$.*

Proof. Without loss of generality, let $\boldsymbol{x}_0$ excite the k modes of A corresponding to $\lambda_1, \cdots, \lambda_r$. Since A is diagonalizable, let $A = U\Lambda U^{-1}$ be its eigen-decomposition and so $\boldsymbol{x}_0$ excite $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k\}$, i.e., $\boldsymbol{x}_0 = \sum_{i=1}^{k} \beta_i \boldsymbol{u}_i$, with $\beta_i \neq 0$. Define $\mathcal{M}(\lambda_i) = \{j : \boldsymbol{u}_j \text{ is the eigenvector corresponding to } \lambda_i\}$, for $i = 1, \cdots, r$. Furthermore, define the r-dimensional subspace,

$$S := \mathrm{span}\left\{\textstyle\sum_{j\in\mathcal{M}(\lambda_1)} \beta_j \boldsymbol{u}_j,\ \ldots, \sum_{j\in\mathcal{M}(\lambda_r)} \beta_j \boldsymbol{u}_j\right\},$$

where the span is taken over the complex field. We prove by induction that $\boldsymbol{x}_t \in S$ for all $t = 1, \cdots, r$. Notice that $\boldsymbol{x}_0 \in S$ and suppose that $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t-1}\} \subset S$; recall from the proof of Proposition 13 that $\boldsymbol{x}_t = A\boldsymbol{z}_{t-1}$, where $\boldsymbol{z}_{t-1} = \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{t-2})^\perp}(\boldsymbol{x}_{t-1})$. Since $\boldsymbol{x}_{t-1} \in S$ and $\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{t-2}\} \subset S$, one can conclude that $\boldsymbol{z}_{t-1} \in S$, and from the definition of S, $\boldsymbol{x}_t = A\boldsymbol{z}_{t-1} \in S$. On the other hand, since $\lambda_1, \lambda_2, \cdots, \lambda_r$ are distinct eigenvalues, $L_k^r(A)$ is full column rank, and thus by Proposition 13, $\mathbf{dim}\,(\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\}) = r$. By hypothesis of the induction $\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\} \subset S$, and since $\mathbf{dim}(S) = r$, we conclude that $\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\}$ must be the entire S, i.e. $\mathrm{span}\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}\} = S$, proving the first claim. Lastly, since $\boldsymbol{x}_r \in S$, $\boldsymbol{z}_r = \mathrm{Proj}_{\mathcal{R}(\mathcal{X}_{r-1})^\perp}(\boldsymbol{x}_r) = 0$, and thus $\boldsymbol{x}_{r+1} = A\boldsymbol{z}_r = 0$, thereby completing the proof. □

Following Theorem 14, if $\boldsymbol{x}_0$ excites k modes of the system corresponding to distinct eigenvalues with trivial algebraic multiplicities, then Algorithm 1 identifies all the excited modes of the system in k iterations. Furthermore, this implies that $\boldsymbol{x}_{k+1} = 0$, i.e., DGR eliminates the unstable modes and regulates the unknown system in exactly $k+1$ iterations.

IV. DGR Example on X-29A

In order to showcase the advantages of our method in a practical setting, we have implemented DGR on data collected from the X-29A aircraft. The Grumman X-29A is an experimental aircraft initially tested for its forward-swept wing; it was designed with a high degree of longitudinal static instability (due to the location of the aerodynamic center on the wings) for maneuverability, where linear models were leveraged to determine the closed-loop stability. We study the case where the dynamics of the aircraft has been perturbed and unknown. Considering the new dynamics (still unstable), we run DGR to regulate the trajectory. The nominal system parameters are obtained from Tables 13-14 in [20] (with a fixed discretization step-size $\gamma = 0.05$), whereas a perturbation ΔA is assumed to shift the dynamics to,

$$\boldsymbol{x}_{t+1} = (A + \Delta A)\boldsymbol{x}_t + B\boldsymbol{u}_t + \boldsymbol{\omega}_t,$$

where every element of ΔA is sampled independently from a normal distribution $\mathcal{N}(0, 0.03)$, and $\boldsymbol{\omega}_t$ denotes the process noise. With no assumptions on the initial controller, we now aim to regulate the new (fixed) unstable system $A_\diamond := A + \Delta A$ in the presence of process noise $\boldsymbol{\omega}_t \sim \mathcal{N}(\mathbf{0}, 0.02\,\mathbf{I})$ and from a random initial condition $\boldsymbol{x}_0 \sim \mathcal{N}(\mathbf{0}, 0.1\,\mathbf{I})$. Note that both the original system and the perturbed system have input characteristics that make them regularizable (with $\rho(\tilde{A}) = 0.994$ and $\rho(\tilde{A}_\diamond) = 0.987$).[9] Without using DGR the norm of the state $\|\boldsymbol{x}_t\|$ would grow rapidly as the unknown system is unstable (with $\rho(A_\diamond) = 1.21$). The resulting state trajectories are demonstrated in Figure 3. As the plot suggests, with DGR in the feedback loop, the unstable modes can be suppressed resulting in stabilization of the system; in general, once enough data has been generated, DGR can be replaced by another (robust) controller for performance. For this example, the bound derived in Theorem 10 is depicted for comparison. The behavior of the bound follows our observations in Remark 11; the bound increases as the algorithm initially tries to "detect" the unstable modes, followed by suppressing these modes for regulation.

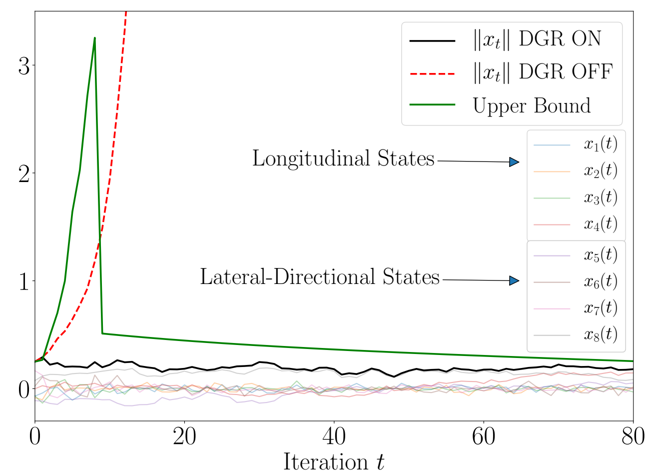

Fig. 3: The state trajectory of X-29 with and without DGR.

V. Concluding Remarks

In this paper, we have introduced the DGR, an online data-parameterized feedback regulator for an unknown and potentially unstable linear system using streaming data from a single trajectory. In addition to its regulation, DGR leads to informative data generation that can subsequently be used for data-guided stabilization or sysID. Along the way, we provided novel system theoretic notions such as "regularizability" and "instability number" in order to analyze the performance and derive bounds on the trajectory of the system over finite-time intervals. Subsequently, we presented the application of the proposed online synthesis procedure on a highly maneuverable unstable aircraft.

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[9] Notation consistent with (3).