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Cooperative, Non-cooperative and Greedy Pursuers Strategies in Multi-Player Pursuit-Evasion Games

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***Abstract*—In this paper we consider three different strategies for N pursuers and one evader in a multi-player Pursuit-Evasion game. Each pursuer's objective function reflects a desire to minimize the distance between itself and the evader while the evader's objective function reflects a need to escape by maximizing the distance between itself and a weighted measure of the distances between itself and the pursuers. The first strategy is characterized by pursuers who cooperate as a team in their effort to catch the evader. The resulting game is referred to as a *Cooperating Pursuers Game*. The second strategy is characterized by non-cooperating pursuers who act in a non-cooperative manner among themselves and the evader. The resulting game is referred to as a *Non-Cooperating Pursuers Game*. The third strategy is characterized by greedy pursuers who act independently and selfishly each on its own in an attempt to catch the evader. The resulting game is referred to as *Greedy Pursuers Game*. To treat these strategies under one common framework a general all-against-one linear quadratic dynamic game is considered and the corresponding closed-loop Nash solution is discussed. Using this framework, the three pursuers' strategies are then developed separately. Implementation of several scenarios of these games are also investigated where neither the pursuers nor the evader have knowledge of the objective functions of the other side and hence need to implement strategies that are secure against possible worst strategies by the other side. A Monte Carlo analysis over the parameters space of the objective functions is preformed to yield probabilities of capture of the evader under each of the studied scenarios. Results of the Monte Carlo simulation show that in general, pursuers do not always benefit from cooperating as a team and that acting as non-cooperating players may yield a higher probability of capturing the evader depending on what strategy the evader may use.**

I. INTRODUCTION

Pursuit-Evasion (PE) refers to a game in which one or more evaders try to escape from a number of pursuers. The solution of these types of games involves the development of movement strategies for all pursuers and evaders as players in a game that will conclude with the evader either escaping or being captured. Problems that include many evaders can be investigated as a combination of resource allocation problems involving multiple pursuers and one evader. That is, a wide range of problems can be covered by studying multi-pursuers one-evader games.

In 1957, *Berge* [1] proposed a PE game in which the evader moves in a prescribed trajectory and the pursuers move with constant velocities but in directions that point directly towards the evader. In 1965, *Isaacs* [2] considered a zero-sum differential game with one pursuer and one evader. Following his work a variety of different formulations of PE games have been studied in the literature [3]–[9] including zero-sum [5], non-zero sum [10] and even Stackelberg [8], [9] games. The dynamics of the game can be formulated either in terms of velocity control [5], [11], or in terms of constant velocities but direction of movement control [12]–[14]. In recent years, multi-pursuers one-evader games have received considerable attention in the literature [3], [7], [15]–[18]. These games present numerous challenges in developing optimal strategies in comparison to the conventional single-pursuer single-evader game; including the structure, existence and uniqueness of the pursuers strategies [19]–[23]. The effectiveness of different strategies in presence of parameter uncertainties for a dynamic two-step look-ahead game has been studied in [24]. In [15], a multi-pursuers one-evader game has been studied in the presence of limited state information for the pursuers' side. The problem is formulated as a 2-player linear-quadratic game with the pursuers cooperating as one team (i.e. as one player) against the evader. This necessitated the introduction of so-called best achievable performance indices for the pursuers. In [25], a two-pursuer one-evader game is considered and decomposition for reduction of time complexity of the solution is introduced and comparing the run-time complexity of both the decomposed and the full game has shown that with increasing cardinality of each player's strategy space the decomposed game yields a relevant decrease of the run-time.

In this paper, and before we investigate multi-pursuers one-evader games, we will consider a general framework that can be used to investigate several possible structural solutions for these types of PE games. More specifically, we consider an (N+1)-player linear quadratic game where N players are minimizing quadratic objective functions while the remaining player is maximizing a similar quadratic objective function. We refer to these games as all-against-one, reflecting the fact that one player's objective is directly opposed to all the other players. In section II we consider a general treatment of all-against-one Linear Quadratic (LQ) games including the derivation of a new sufficient conditions for exponential stability of the closed-loop Nash solutions. In section III we formulate the N-pursuers one-evader game as a special case of the all-

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against-one game and investigate three different strategies for the pursuers. These strategies include non-cooperating pursuers, cooperating pursuers and greedy pursuers. In section IV, we present several illustrative examples that describe different pursuit-evasion scenarios and show simulation results for all three pursuers' strategies when the evader is using a strategy that yields the Nash equilibrium in each case. In section V, we present some preliminary results where neither the pursuers nor the evader have knowledge about the objective functions of the other side and hence need to implement strategies that are secure against possible strategies by the other side. Results of Monte Carlo simulations of these incomplete information games are also reported. Concluding remarks are presented in section VI.

II. All-Against-One Games

Consider an all-against-one game with $M = N + 1$ players where player 1 is the opposing player and players 2 through M are all against player 1. The concept of all-against-one games does not only apply to pursuit-evasion problems, but it can also be utilized to analyze large systems consisting of many subsystems (System of Systems) where one subsystem decides to dissent from the remaining subsystems. The remaining subsystems become an all-against the dissenting subsystem [26]. To make the problem mathematically tractable, let the game be LQ with dynamics in the standard form [10] and [27]

$$\dot{z} = Az + B_1 u_1 + \sum_{j=2}^{M} B_j u_j, \quad z(t_0) = z_0; \tag{1}$$

where z is the state vector and u_1 through u_M are the player's control input vector. Matrix A is the open-loop system dynamics, matrices B_1 through B_M are player's input matrices, respectively. Let the objective function of player 1 be of the form

$$J_1 = \frac{1}{2} z(t_f)^\intercal S_{1f} z(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [z(t)^\intercal Q_1 z(t) + u_1(t)^\intercal R_1 u_1(t)] dt, \tag{2}$$

where S_{1f} and Q_1 are symmetric negative definite matrices[1] and R_1 is symmetric positive definite matrix. Let the objective functions of players 2 through M be of the form

$$J_i = \frac{1}{2} z(t_f)^\intercal S_{if} z(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [z(t)^\intercal Q_i z(t) + u_i(t)^\intercal R_i u_i(t)] dt; \tag{3}$$

for $i = 2, 3, ..., M$, where S_{if} and Q_i are symmetric positive semi-definite matrices and R_i is symmetric positive definite matrix. The main difference between this game and the standard LQ game considered in the literature [28]–[30] is that player 1 has weight matrices S_{1f} and Q_1 that are not of the same sign as similar matrices of the other players. In other words, player 1 is trying to unregulate (drive the state vector of the system away from the origin) while players 2 through M are trying to regulate the system (drive the state vector of the system to the origin). It is well known that necessary conditions for the closed-loop Nash strategies for this game are of the form [10]

$$u_i^* = -R_i^{-1} B_i S_i z \tag{4}$$

for $i = 1, 2, ..., M$, where S_i's satisfy the following M-coupled differential Riccati equations,

$$\dot{S}_i + S_i A + A^\intercal S_i + Q_i + S_i B_i R_i^{-1} B_i^\intercal S_i - \sum_{j=1}^{M} (S_i B_j R_j^{-1} B_j^\intercal S_j + S_j B_j R_j^{-1} B_j^\intercal S_i) = 0, \tag{5}$$

where $S_i(t_f) = S_{if}$.

Next, we present a theorem that provides sufficient conditions for uniform exponential stability and assures that all states of the system are within a defined hyperball at the end of the game. Later in section IV, this result is used to guarantee capture of the evader.

Theorem 1. *For the all-against-one game, let the implemented strategy for player $1, 2, \ldots, M$ be as (4) and define*

$$C(t) \triangleq \sum_{j=1}^{M} (Q_j + S_j B_j R_j^{-1} B_j^\intercal S_j) \tag{6}$$

and

$$P(t) \triangleq \sum_{j=1}^{M} S_j(t). \tag{7}$$

where $S_i(t)$ is solution of (5) for $t \in [t_0, t_f]$. If $C(t) > 0$ for all $t \in [t_0, t_f]$ and $P(t_f) \geq 0$, then the system will be uniformly exponentially stable and $P(t) > 0$ for all $t \in [t_0, t_f)$. Furthermore, for any positive scalar $\delta > 0$, if

$$t_f \geq t_0 + \frac{\bar{\lambda}_P}{\underline{\lambda}_C} \ln\left(\frac{\bar{\lambda}_P . \|z_0\|^2}{\underline{\lambda}_P . \delta^2}\right) \tag{8}$$

then it follows that $\|z(t_f)\| \leq \delta$.

In the above theorem, $\bar{\lambda}_{(.)} = \max_{t\in[t_0,t_f]} \lambda_{(.)}^{max}$ and $\underline{\lambda}_{(.)} = \min_{t\in[t_0,t_f]} \lambda_{(.)}^{min}$ where $\lambda_{(.)}^{max}$ and $\lambda_{(.)}^{min}$ are the maximal and minimal eigenvalues. Also, $\|.\|$ refers to the Euclidean norm.

Proof. Let

$$\bar{A} = A - \sum_{j=1}^{M} B_j R_j^{-1} B_j^\intercal S_j, \tag{9}$$

following (5), $S_i(t)$ should satisfy

$$\dot{S}_i + S_i \bar{A} + \bar{A}^\intercal S_i + Q_i + S_i B_i R_i^{-1} B_i^\intercal S_i = 0; \tag{10}$$

for $i = 1, 2, 3, ..., M$, and $S_i(t_f) = S_{if}$. Let $\Phi(t, \tau)$ be the state transition matrix of $-\bar{A}^\intercal(t)$. It is known that for $t, \tau \in [t_0, t_f]$,

$$\frac{d\Phi(t,\tau)}{dt} = -\bar{A}^\intercal(t)\Phi(t,\tau), \quad \Phi(\tau,\tau) = I; \tag{11}$$

[1] We require these matrices to be negative definite instead of negative semi-definite so that player 1 would not have the option to exclude any of the players who are against it.

The matrix $P(t)$ satisfies the differential equation

$$\dot{P} + \bar{A}^{\mathsf{T}} P + P\bar{A} + C(t) = 0, \quad P(t_f) = P_{t_f}; \tag{12}$$

where $P_{t_f} = \sum_{j=1}^{M} S_{jf}$. Consequently $P(t)$ also satisfies

$$P(t) = \Phi(t, t_f) P(t_f) \Phi^{\mathsf{T}}(t, t_f) + \int_t^{t_f} \Phi(t, \tau) C(\tau) \Phi^{\mathsf{T}}(t, \tau) d\tau.$$

Since $P(t_f) \geq 0$ and $C(t) > 0$, it follows that $P(t) > 0$, for all $t \in [t_0, t_f)$. Now we can define the Lyapunov function

$$V(t) = z^{\mathsf{T}} P(t) z, \tag{13}$$

then

$$\begin{aligned} \dot{V}(t) &= \dot{z}^{\mathsf{T}} P(t) z + z^{\mathsf{T}} \dot{P}(t) z + z^{\mathsf{T}} P(t) \dot{z} && (14) \\ &= z^{\mathsf{T}} (\bar{A}^{\mathsf{T}} P + \dot{P} + P\bar{A}) z && (15) \\ &= -z^{\mathsf{T}} C(t) z && (16) \end{aligned}$$

So $C(t) > 0$ results in uniform exponential stability of the system. Additionally, since

$$\dot{V} \leq -\frac{\lambda_C}{\bar{\lambda}_P} z^{\mathsf{T}} P z \leq -\frac{\lambda_C}{\bar{\lambda}_P} V \tag{17}$$

then one can conclude that

$$V(t) \leq V(t_0) \exp[-\frac{\lambda_C}{\bar{\lambda}_P}(t - t_0)]. \tag{18}$$

Moreover,

$$\begin{aligned} \|z(t)\|^2 &\leq \frac{1}{\underline{\lambda}_P} V(t) && (19) \\ &\leq \frac{V(t_0)}{\underline{\lambda}_P} \exp[-\frac{\lambda_C}{\bar{\lambda}_P}(t - t_0)] && (20) \\ &\leq \frac{\bar{\lambda}_P}{\underline{\lambda}_P} \|z_0\|^2 \exp[-\frac{\lambda_C}{\bar{\lambda}_P}(t - t_0)]. && (21) \end{aligned}$$

Finally, in order to achieve $\|z(t_f)\| \leq \delta$, t_f should satisfy

$$\frac{\bar{\lambda}_P}{\underline{\lambda}_P} \|z_0\|^2 \exp[-\frac{\lambda_C}{\bar{\lambda}_P}(t_f - t_0)] \leq \delta^2 \tag{22}$$

which yields (8) and this ends the proof. □

Remark 1. The above theorem has been proved for M players with different quadratic cost functions in a linear game while they are implementing their Nash strategies. Later in Section III, different Nash solutions will be presented in which players either utilizes their Nash strategies according to an M-player (all-against-one) game or 2-player (greedy or one-against-one) games or cooperative (one-team-against-one) game. Thus, the above theorem can be applied directly to each of those scenarios.

The following remark represents usefulness of the above theorem by providing a practical explicit sufficient condition resulting in exponential stability of the system.

Remark 2. For all-against-one games, let the implemented strategies for player $1, 2, \ldots, M$ be as (4). Noting that S_{1f} and Q_1 are negative semi-definite matrices, if $S_{1f} + \sum_{j=2}^{M} S_{jf} \geq 0$ and $Q_1 + \sum_{j=2}^{M} Q_j \geq 0$ for all $t \in [t_0, t_f]$ then the system

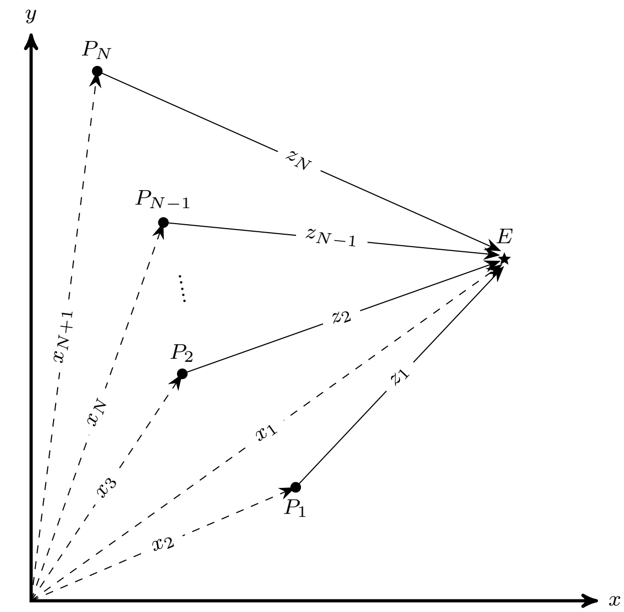


Fig. 1. Pursuers and the evader on an x-y coordinate system

is exponentially stable regardless of the initial state vector z_0, t_0 and t_f. In other words, if the collective weights of players 2 through M overpower the weights of player 1 (opposing player) in the objective functions then the exponential stability of the entire system (1) is assured.

Remark 3. As it is shown in the second part of the theorem, if conditions (6) and (7) are satisfied and if termination time t_f (8) is large enough, then the state vector of the system is guaranteed to terminate in a small hyperball. In the next section, this means that capturing of the evader is guaranteed no matter how far the pursuers are at t_0.

III. Pursuit-Evasion Game Formulation

Following the all-against-one game formulation in Section II, consider a PE game with N pursuers against one evader. The game is formulated as a finite time LQ game in which the goal of the i-th pursuer is to minimize its distance with respect to the evader while the evader's goal is to maximize a conical combination of its distances from the pursuers. In this section we are concerned with developing only strategies for the pursuers. In doing so, the pursuers will assume that the evader will be using the corresponding Nash strategy but will not know whether the evader will implement that strategy or not.

Let $x_1 \in \mathbb{R}^n$ be the evader's position vector and $x_{i+1} \in \mathbb{R}^n$ be i-th pursuer's position vector and let

$$z_i = x_1 - x_{i+1}, \tag{23}$$

for $i = 1, 2, ..., N$, be the difference between position vector of the i-th pursuer and the evader as illustrated in Figure 1. The system dynamics follows (1), where the state vector z is defined as $z = \begin{bmatrix} z_1^{\mathsf{T}} & z_2^{\mathsf{T}} & \ldots & z_N^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ and $u_i \in \mathbb{R}^{n \times 1}$. Since the pursuers' dynamics are uncoupled, the system dynamics matrix $A = blkdiag\{A_1, ..., A_N\} \in \mathbb{R}^{nN \times nN}$ is block diagonal where $A_i \in \mathbb{R}^{n \times n}$ for $i = 1, 2, ..., N$. The input matrix of the evader is $B_e = B_1 = \mathbf{1}_N \otimes \mathbf{I}_n \in \mathbb{R}^{nN \times n}$ and the input matrices of the pursuers are $B_{p_i} = B_{i+1} = -\mathbf{e}_i \otimes \mathbf{I}_n \in \mathbb{R}^{nN \times n}$ for $i = 1, 2, ..., N$. $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ is the identity matrix, $\mathbf{e}_i \in \mathbb{R}^N$ denotes the vector with a 1 in the i-th coordinate and 0's elsewhere, $\mathbf{1}_N \in \mathbb{R}^{N \times 1}$ is a vector with all the entries equal to 1 and $\otimes$ is the Kronecker product. Dimension n can be any positive integer, for example 2 or 3 which represents a PE game with 2 or 3 dimensions, respectively. Each player should minimize the objective function as defined in (2) and (3), where $S_{if} \in \mathbb{R}^{nN \times nN}, Q_i \in \mathbb{R}^{nN \times nN}$ and $R_i \in \mathbb{R}^{n \times n}$ are the dimensions of weight matrices of the i-th player. For the player 1, the evader, $Q_e = Q_1 < 0$, $S_{e_f} = S_{1f} < 0$ and $R_e = R_1 > 0$. For players 2 thorough M, the pursuers, $Q_{p_i} = Q_{(i+1)} \geq 0$, $S_{p_i f} = S_{(i+1)f} \geq 0$ and $R_{p_i} = R_{(i+1)} > 0$ for $i = 1, 2, ..., N$.

In this paper, we consider three different strategies for the pursuers which have different Nash attributes as illustrated in Figure 2. These strategies are discussed in the following subsections. The weights in the players' objective functions (2) and (3) are chosen as follow

$$\begin{aligned} S_{p_{fi}} &= diag\{f_{p_i}\mathbf{e}_i\} \otimes I_n, & S_{e_f} &= -\,diag\{f_{e_1}, ..., f_{e_N}\} \otimes I_n, \\ Q_{p_i} &= diag\{q_{p_i}\mathbf{e}_i\} \otimes I_n, & Q_e &= -\,diag\{q_{e_1}, ..., q_{e_N}\} \otimes I_n, \\ R_{p_i} &= r_{p_i} \otimes I_n, & R_e &= r_e \otimes I_n, \end{aligned} \tag{24}$$

where f_{e_i}, q_{e_i}, r_e and $f_{p_i}, q_{p_i}, r_{p_i}$ for $i = 1, 2, ..., N$ are positive scalar weights for the evader and corresponding i-th pursuer, respectively. Furthermore, for simplicity, we will assume that $t_0 = 0$.

We should also mention that in developing the pursuers' strategies we will assume that all players have full measurements of the state vectors they need, to implement their closed-loop strategies. In practice, however, full state measurements may not be available. Problems with partial state measurement and problems with information network topology among the players will be considered in a follow-up paper.

A. Cooperative Pursuers' Strategy

This game describes a situation where the pursuers act as one team against the evader as illustrated in Figure 2a. Essentially, it is a 2-player game where all the pursuers' controls are combined as one control vector. The cooperation among the pursuers can be formulated as a convex combination of their cost functions as $J_P = \sum_{i=1}^N \alpha_i J_{p_i}$, where $\alpha_i > 0$, $\sum_{i=1}^N \alpha_i = 1$ and $J_{p_i} = J_{i+1}$ for $i = 1, 2, ..., N$ as they are defined in (2) and (3). The weight α_i can be interpreted as the fraction of the total effort that the i-th pursuer is contributing towards the team. Consequently, the cost function for the pursuers as a team becomes

$$J_P = \frac{1}{2} z(t_f)^\intercal S_{P_f} z(t_f) + \frac{1}{2}\int_0^{t_f} [z(t)^\intercal Q_P z(t) + u_i(t)^\intercal R_P u_i(t)]dt \tag{25}$$

where

$$S_{P_f} = \sum_{i=1}^N \alpha_i S_{p_{fi}},\ Q_P = \sum_{i=1}^N \alpha_i Q_{p_i},\ R_P = \sum_{i=1}^N \alpha_i R_{p_i}.$$

The evader's cost function as defined in (2) becomes $J_e = J_1$. The system dynamics of (1) becomes

$$\dot{z} = Az + B_e u_e + B_P u_P, \tag{26}$$

where $u_e = u_1 \in \mathbb{R}^{n \times 1}$ is the evader's control vector and $u_P = [u_2^\intercal, u_3^\intercal, ..., u_M^\intercal]^\intercal \in \mathbb{R}^{Nn \times 1}$ is the combined control vector of the pursuers as a team. The matrix $B_e = \mathbf{1}_N \otimes \mathbf{I}_n$ is the input matrix of the evader and $B_P = -\mathbf{I}_N \otimes \mathbf{I}_n$ is the combined input matrix of pursuers as a team.

The system dynamics (26) and objective functions J_P and J_e form a 2-player LQ game. A Pareto Front solution of cooperation among the pursuers can be obtained by using different values of α_i which impacts the performance of the pursuers as will be illustrated in Section V. The closed-loop Nash strategies for the pursuers can be obtained from (4) as follows

$$u_P^* = -R_P^{-1} B_P S_P z, \tag{27}$$

$$u_e^* = -R_e^{-1} B_e S_e z; \tag{28}$$

where the matrices S_P and S_e are the solutions to the following coupled differential Riccati equations

$$\dot{S_P} + S_P A + A^\intercal S_P + Q_P - S_P B_P R_P^{-1} B_P^\intercal S_P - S_P B_e R_e^{-1} B_e^\intercal S_e - S_e B_e R_e^{-1} B_e^\intercal S_P = 0, \tag{29}$$

$$\dot{S_e} + S_e A + A^\intercal S_e + Q_e - S_e B_e R_e^{-1} B_e^\intercal S_e - S_e B_P R_P^{-1} B_P^\intercal S_P - S_P B_P R_P^{-1} B_P^\intercal S_e = 0, \tag{30}$$

where $S_P(t_f) = S_{P_f}$ and $S_e(t_f) = S_{e_f}$. We note that the evader's strategy (28) is only calculated in order to determine the pursuers' strategy, but there is no guarantee that the evader will actually use that strategy.

B. Non-Cooperative Pursuers' Strategies

This game represent a situation where the pursuers are competing among themselves in their effort to capture the evader as illustrated in Figure 2b. The Nash equilibrium in this case will involve all pursuers and evader simultaneously, and the corresponding non-cooperative pursuers' strategies can be determined using (4) and (5).

C. Greedy Pursuers' Strategies

This game represent a situation where the pursuers are greedy. That is, each pursuer decides to ignore all other pursuers and pursue the evader on its own as illustrated in Figure 2c. Essentially, in this case we have N separate PE games between each pursuer and the evader. The pursuers' strategies are similar to the ones proposed in [1] where each pursuer moves exactly toward the evader at each instant of time, ignoring what the others are doing.

According to the problem formulation (23) and (24), dynamics of the games can be represented by N decoupled systems

$$\dot{z}_i = A_i z_i + B_{e_i} u_{e_i} + B_{p_i} u_{p_i}, \tag{31}$$

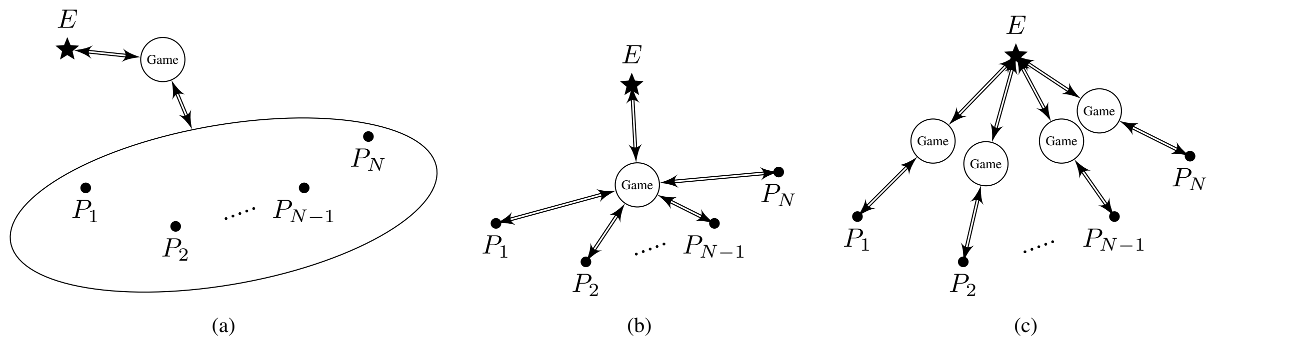


Fig. 2. (a) PE game between cooperative pursuers (as a team) and the evader, (b) PE game among non-cooperative pursuers and the evader, (c) Separate PE games between each greedy pursuer and the evader.

for $i = 1, 2, ..., N$, where $B_{e_i} = B_e \circ (\mathbf{e}_i \otimes \mathbf{I}_n)$ is the corresponding part of the original input matrix B_e as defined in (3) with respect to i-th player, with input matrix $B_{p_i} \in \mathbb{R}^{n\times n}$, and $\circ$ is the Hadamard product. If each pursuer ignores the effect of the other pursuers in its game with the evader, it assumes that the evader is playing only versus itself and ignoring the others. In another word, the evader's cost function against i-th pursuer is as follows

$$J_{e_i} = \frac{1}{2} z(t_f)^\intercal S_{e_i f} z(t_f) + \frac{1}{2}\int_0^{t_f} [z(t)^\intercal Q_{e_i} z(t) + u_{e_i}(t)^\intercal R_e u_{e_i}(t)]dt, \quad (32)$$

where $S_{e_i f} = S_{e_f} \circ blkdiag\{\mathbf{e}_i \otimes \mathbf{I}_n\}$ and $Q_{e_i} = Q_e \circ blkdiag\{\mathbf{e}_i \otimes \mathbf{I}_n\}$ are the evader weights instead of S_{e_f}, Q_e in (24), respectively. According to (4) and (5), the closed-loop Nash greedy strategies can be obtained for each pursuer as follows

$$u^*_{p_i g} = -R_{p_i}^{-1} B_{p_i} S_{p_{fi}} z, \quad (33)$$

$$u^*_{e_i g} = -R_e^{-1} B_{e_i} S_{e_i} z; \quad (34)$$

where $S_{p_{fi}}$ and S_{e_i} are the solutions to the following 2-coupled differential Riccati equations

$$\dot{S}_{p_i} + S_{p_i}\bar{A}_i + \bar{A}_i^\intercal S_{p_i} + Q_{p_i} - S_{p_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^\intercal S_{p_i} - S_{p_i} B_{e_i} R_e^{-1} B_{e_i}^\intercal S_{e_i} - S_{e_i} B_{e_i} R_e^{-1} B_{e_i}^\intercal S_{p_i} = 0 \quad (35)$$

$$\dot{S}_{e_i} + S_{e_i}\bar{A}_i + \bar{A}_i^\intercal S_{e_i} + Q_{e_i} - S_{e_i} B_{e_i} R_e^{-1} B_{e_i}^\intercal S_{e_i} - S_{e_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^\intercal S_{p_i} - S_{p_i} B_{p_i} R_{p_i}^{-1} B_{p_i}^\intercal S_{e_i} = 0 \quad (36)$$

for each $i = 1, 2, ..., N$, where $S_{p_i}(t_f) = S_{p_i f}$, $S_{e_i}(t_f) = S_{e_i f}$ and matrix $\bar{A}_i = blkdiag\{\mathbf{e}_i \otimes A_i\}$. Note that equation (33) is in form of full state closed-loop. However, $S_{p_{fi}}$ contains zeros in all entries except the ones related to i-th pursuer, indicating that the i-th pursuer needs only measurement of the distance between itself and the evader to implement its control.

IV. Illustrative Example

Consider a three-pursuer one-evader game on the plane (so $n = 2$) with parameters as described in Table I. The entries under the position column are the x-y coordinates of each player, under the f column represent the weights in S_{if}, under the r column represent the weights in R_i. The parameters in the evader row are the same against each of three pursuers. For simplicity, the control input of each player is assumed to be velocity which leads to a system dynamics matrix of $A = \mathbf{0}$. The terminal time is assumed to be $t_f = 5$ and the evader is assumed to be captured if it enters a capture circle of radius $\delta = 0.1$ from any one pursuer. The three pursuers' strategies are determined as described in the previous section and the evader in each case is implementing the corresponding Nash strategy in each case. The resulting trajectories and corresponding distances of pursuers to the evader are illustrated in Figure 3 and 4, respectively. In the cooperative pursuers game, $\alpha_i = 0.\bar{3}$ are used in the team objective functions, representing equal contributions by all pursuers. Note that the evader is able to escape when

TABLE I
Positions and Parameters of Player in the Example

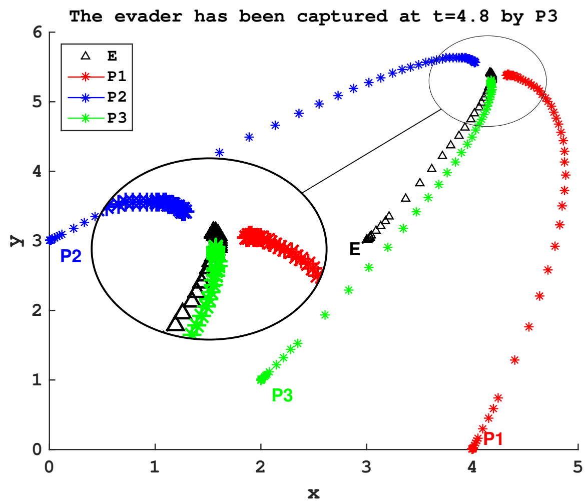
	Position	Parameters[1]		
		f	q	r
Evader	$[3, 3]$	-3	-3	2
Pursuer 1	$[4, 0]$	3.1	3.1	7.5
Pursuer 2	$[0, 3]$	3.1	3.1	10.5
Pursuer 3	$[2, 1]$	4	4	15

1. Parameters are the diagonal entries in (24).

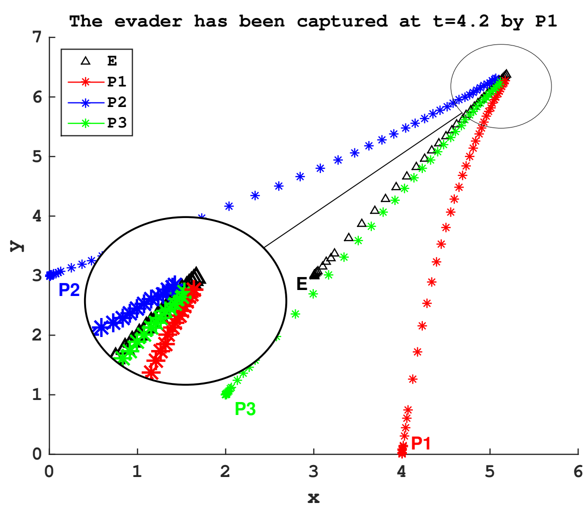
each pursuer uses the greedy strategy but it gets captured when the pursuers implement either the cooperative or the non-cooperative strategies. However, the method by which the evader gets captured in these two games are noticeably different. When the pursuers cooperate as a team they seem to encircle and entrap the evader before capturing (Figure 3a) while when they do not cooperate they seem to target the evader directly (Figure 3b). In Figure 3b, while the players are implementing the Non-cooperative Nash strategies as (4), the conditions of Remark 2 are satisfied as follows

$$S_{1f} + \sum_{j=2}^{4} S_{jf} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes I_2 \geq 0$$

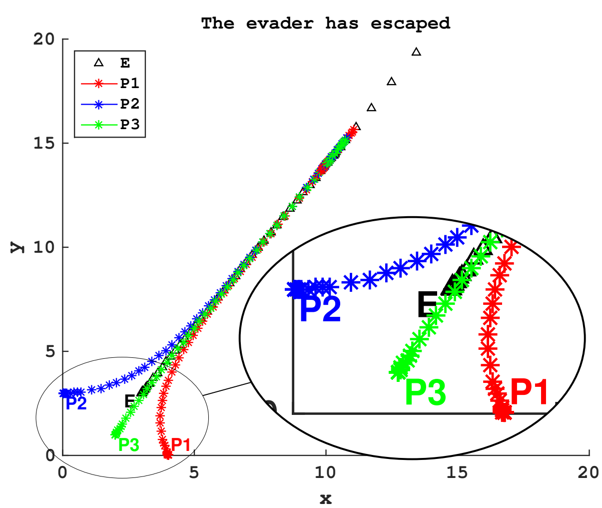
and similarly $Q_1 + \sum_{j=2}^{M} Q_j \geq 0$. Consequently, the exponential stability of the system is guaranteed regardless of the initial position of the players. In the greedy pursuers'



(a)

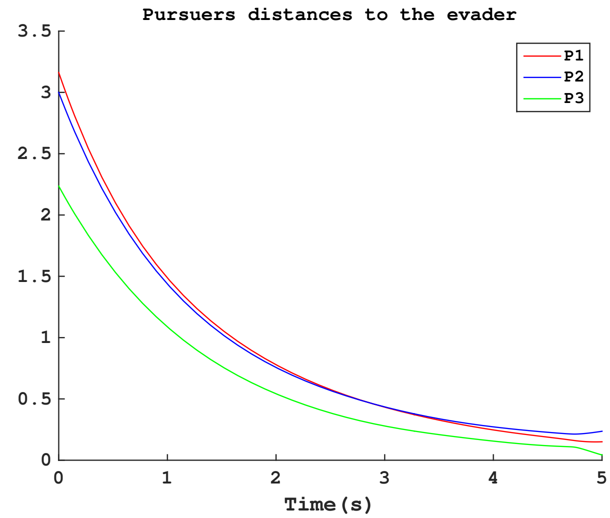


(b)

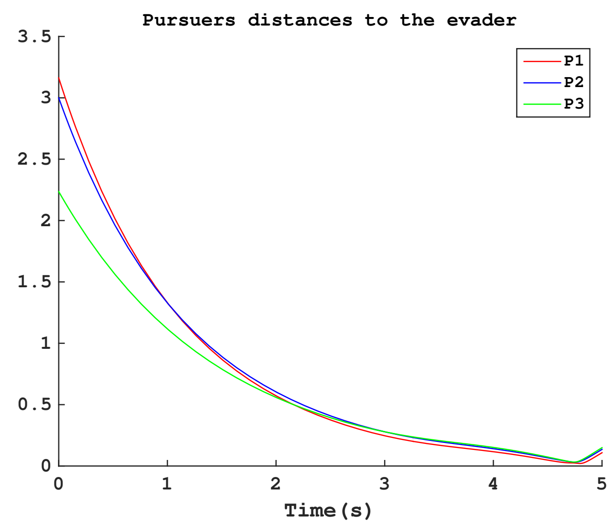


(c)

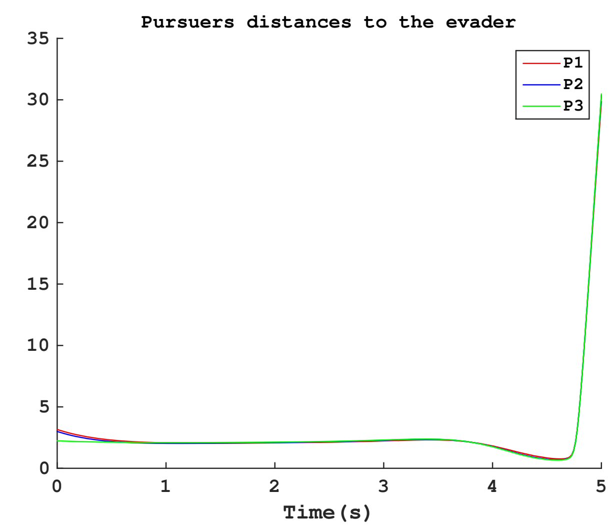
Fig. 3. Example - Trajectories of players for (a) Cooperative pursuers, (b) Non-cooperative pursuers and (c) Greedy pursuers.



(a)



(b)



(c)

Fig. 4. Example - Corresponding distances of pursuers to the evader for (a) Cooperative pursuers, (b) Non-cooperative pursuers and (c) Greedy pursuers.

strategy, the fact that each pursuers are moving exactly towards the evader causes the evader to escape (Figure 3c). In this case the condition of Theorem 1 are not satisfied. We should note that in this example we have assumed that all players have complete information about parameters of the other side, which also may not be feasible in realistic situations.

V. Simulation

In the illustrative example in the previous section, we have assumed that each player has full knowledge of the parameters in the objective functions of all other players (i.e. Table I). In a practical PE game, however, this may not the case. We have also assumed that in each of the three pursuers' strategies considered, the evader is implementing the strategy that yields a Nash equilibrium in each case. In a practical PE game, this also may not be the case.

With such lack of knowledge of the game parameters and strategy used by the evader, the pursuers might be interested in determining the probabilities of capturing the evader under some statistical distribution of these parameters and for several possible strategies used by the evader. To accomplish this, we performed a Monte Carlo simulation on a 3-pursuers one-evader scenario with parameter distributions as shown in Table II. In the cooperative pursuers' strategies we used four different team configurations with different choices of α_i parameters as shown in Table III. In Team 1, 2 and 3 we have given one pursuer a higher share of the total effort (0.8) than the other two and in Team 4 we have assigned equal share of the total effort (0.3) to the three pursuers. Each sample of the MC simulation represents all different combinations of strategies for the pursuers and evader. The outcome of each simulation is defined to be a Bernoulli random variable of 1 if the pursuers capture the evader with probability of (ρ) and of 0 if the evader escapes with probability of ($1-\rho$). Assuming that the error distribution is normal as in [31], [32], the confidence interval of the simulated mean value would be $[\rho - z_{\gamma/2}.s,\ \rho + z_{\gamma/2}.s]$, where $s = \sqrt{\frac{\rho(1-\rho)}{n}}$ is the standard deviation of n samples and $z_{\gamma/2}$ is the critical value of the Normal distribution for a given error level γ. The desired margin of error ε is half of the confidence interval, so the number of required samples can be calculated as follows:

$$n = \left[\frac{z_{\gamma/2}\sqrt{\rho(1-\rho)}}{\varepsilon}\right]^2. \tag{37}$$

For a 95% confidence level, the error level is $\gamma = 0.05$ and the critical value would be $z_{\gamma/2} = 1.96$. For a desired margin of error $\varepsilon = 0.01$, the resulting minimum number of required samples becomes $9,604$ which happens when $\rho = 0.5$. Consequently, in the worst case, with a confidence level of 95% we can assert that the percentages of capturing probability will be accurate with a maximum error of $\pm 1\%$. In our Monte Carlo simulation we used $10,000$ runs, each consisting of all possible combinations of strategies by pursuers and evader. This required 15 hours of CPU time on a super computer with 10 Intel Xeon 64-bit CPUs. The results of the simulation are shown in Table IV.

TABLE II
Positions and Parameters of Pursuers in MC Simulation

	Position	Parameters f	Parameters q	Parameters r
Evader	$[3,3]$	$U[-3,-1]^\star$	$U[-3,-1]$	$U[1,3]$
Pursuer 1	$[4,0]$	$U[0,2]$	$U[0,2]$	$U[1,10]$
Pursuer 2	$[0,3]$	$U[0,2]$	$U[0,2]$	$U[1,10]$
Pursuer 3	$[2,1]$	$U[0,2]$	$U[0,2]$	$U[1,10]$

$\star$ $U[a,b]$ indicates the Uniform distribution between a and b.

TABLE III
Value of α_i for Each Pursuer in Different Team Formations

	Pursuer 1 α_1	Pursuer 2 α_2	Pursuer 3 α_3
Team 1	0.8	0.1	0.1
Team 2	0.1	0.8	0.1
Team 3	0.1	0.1	0.8
Team 4	$0.\bar{3}$	$0.\bar{3}$	$0.\bar{3}$

For each possible strategy of the evader, the pursuers strategy that yields the highest probability of capturing the evader is indicated by $\star$ on Table IV. The highest such probability occurs when the pursuers are using the Non-cooperative strategy and the evader uses a Nash Strategy against Team 3. The lowest capturing probability occurs when the pursuers use also the Non-cooperative strategy and the evader uses a Nash strategy against the closest pursuer. In between these two extremes, the pursuers could run the risk of attaining very low probabilities of capturing the evader. For example, if the pursuers cooperate according to Team 3 to achieve a capturing probability of 67%, they run the risk of attaining 2% probability of capture (or 98% of probability of escape) if the evader uses a Nash strategy against the closest pursuer. Thus, the safest strategy for the pursuers is to use the Non-cooperative strategy.

Finally, we should also note that the evader may perform a similar Monte Carlo simulation to arrive at Table IV. From the evader's perspective, for every strategy that the pursuers use, the strategy that yields the lowest probability of capture is indicated by †. Consequently the safest strategy for the evader is to use the Nash strategy against the closest pursuer. If both sides use the safest strategy, this eventually would result in a 46% probability of capture (or 54% probability of escape) of the evader.

VI. Conclusion

In this paper we considered an all-against-one LQ game and provided necessary conditions for the closed-loop Nash strategies as well as sufficient condition for uniform exponential stability of the system and resulting state trajectory. This all-against-one game was used as the framework to study three different pursuers strategies in a multi-pursuers one-evader PE game. However, this framework is not restricted to PE games and it can be easily applied to any type of multi-agent systems

TABLE IV
MONTE CARLO SIMULATION: CAPTURING PROBABILITY IN PERCENT

		Evader's Nash Strategies versus:					
		Team 1	Team 2	Team 3	Team 4	NC Pursuers	Closest Pursuer
Pursuers' Nash Strategies vs Evader: Cooperative Pursuers	Team 1	64	21	29	$61^{\star}$	$20^{\dagger}$	23
Pursuers' Nash Strategies vs Evader: Cooperative Pursuers	Team 2	60	32	$15^{\dagger}$	56	17	20
Pursuers' Nash Strategies vs Evader: Cooperative Pursuers	Team 3	$67^{\star}$	9	8	7	32	$2^{\dagger}$
Pursuers' Nash Strategies vs Evader: Cooperative Pursuers	Team 4	23	28	36	$15^{\dagger}$	25	17
Pursuers' Nash Strategies vs Evader	NC Pursuers	59	$69^{\star}$	$74^{\star}$	53	$48^{\star}$	$\mathbf{46}^{\star\dagger}$
Pursuers' Nash Strategies vs Evader	Greedy Pursuers	18	10	11	12	$7^{\dagger}$	31

NC is Non-Cooperative

in presence of an adversary agent. The strategies considered include cooperative, non-cooperative and greedy pursuers. In each case, these strategies are derived assuming that the evader is also using the corresponding Nash strategy. An example is used to illustrate these strategies and show several different approaches that may be used by the pursuers to capture the evader. Finally, in the absence of complete information on the parameters in the objective functions, we performed a Monte Carlo simulation with 10,000 runs to produce probabilities of capture of the evader under uniform distribution of the parameter spaces and for several different strategies that could be used by the evader against the three proposed strategies for the pursuers.

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